Researches on the Vibration of Pendulums in Fluid Media. By GEORGE GREEN, Esq. Communicated by Sir EDWARD FFRENCH BROMHEAD, Bart. M. A. F. R. SS. Lond. & Ed.

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PROBABLY no department of Analytical Mechanics presents greater difficulties than that which treats of the motions of fluids; and hitherto the success of mathematicians therein has been comparatively limited. In the theory of the waves, as presented by MM. POISSON and CAUCHY, and in that of sound, their success appears to have been more complete than elsewhere; and if to these investigations we join the researches of LAPLACE concerning the tides, we shall have the principal important applications hitherto made of the general equations upon which the determination of this kind of motion depends. The same equations will serve to resolve completely a particular case of the motion of fluids, which is capable of a useful practical application; and, as I am not aware that it has yet been noticed, I shall endeavour, in the following paper, to consider it as briefly as possible.

In the case just alluded to, it is required to determine the circumstances of the motion of an indefinitely extended non-elastic fluid, when agitated by a solid ellipsoidal body, moving parallel to itself, according to any given law, always supposing the body's excursions very small, compared with its dimensions. From what will be shown in the sequel, the general solution of this problem may very easily be obtained. But as the principal object of our paper is to determine the alteration produced in the motion of a pendulum by the action of the surrounding medium, we have insisted more particularly on the case where the ellipsoid moves in a right line parallel to one of its axes, and have thence proved, that, in order to obtain the correct time of a pendulum's vibration, it will not be sufficient merely to allow for the loss of weight caused by the fluid medium, but that it will likewise be requisite to conceive the density of the body augmented by a quantity proportional to the density of this fluid. The value of the quantity last named, when the body of the pendulum is an oblate spheroid, vibrating in its equatorial plane, has been completely determined, and, when the spheroid becomes a sphere, is precisely equal to half the density of the surrounding fluid. Hence, in this last case, we shall have the true time of the pendulum's vibration, if we suppose it to move *in vacuo,* and then simply conceive its mass augmented by half that of an equal volume of the fluid, whilst the moving force with which it is actuated is diminished by the whole weight of the same volume of fluid.

We will now proceed to consider a particular case of the motion of a non-elastic fluid over a fixed obstacle of ellipsoidal figure, and thence endeavour to find the correction necessary to reduce the observed length of a pendulum vibrating through exceedingly small arcs in any indefinitely extended medium to its true length *in vacuo,* when the body of the pendulum is a solid ellipsoid. For this purpose, we may remark, that the equations of the motion of a homogeneous non-elastic fluid are

$$V-\frac{p}{\rho}=\frac{d\phi}{dt}+\tfrac{1}{2}\left\{\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\left(\frac{d\phi}{dz}\right)^2\right\} \quad \text{(1.)}$$

$$0=\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2} \quad \text{(2.)}$$

Vide *Mec. Cel.* Liv. iii. Ch. 8. No. 33, where ϕ is such a function of the co-ordinates x, y, z of any particle of the fluid mass, and of the time t that the velocities of this particle in the directions of and tending to increase the co-ordinates x, y and z shall always be represented by $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$, and $\frac{d\phi}{dz}$ respectively. Moreover, ρ represents the fluid's density, p its pressure, and V a function dependent upon the various exterior forces which act upon the fluid mass.

When the fluid is supposed to move over a fixed solid ellipsoid, the principal difficulty will be so to satisfy the equation (2.) that the particles at the surface of this solid may move along this surface, which may always be effected by making

$$\phi = \left(\lambda + \mu \int_{\infty} \frac{df}{a^3 bc}\right) x\,^* ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.)$$

supposing that the origin of the co-ordinates is at the centre of the ellipsoid: λ and μ being two arbitrary quantities constant with regard to the variables x, y, z; and a, b, c, f being functions of these same variables, determined by the equations

$$a^2 = a'^2 + f, \quad b^2 = b'^2 + f, \quad c^2 = c'^2 + f, \quad \text{and} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; \quad \ldots\ldots\ldots \quad (4.)$$

in which a', b', c' are the axes of the given ellipsoid.

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* In my memoir on the Determination of the exterior and interior Attractions of Ellipsoids of Variable Densities, recently communicated to the Cambridge Philosophical Society by Sir EDWARD FFRENCH BROMHEAD, Baronet, I have given a method by which the general integral of the partial differential equation

$$0 = \frac{d^2 V}{dx_1^2} + \frac{d^2 V}{dx_2^2} + \ldots\ldots\ldots + \frac{d^2 V}{dx_s^2} + \frac{d^2 V}{du^2} + \frac{n - s}{u} \frac{dV}{du}$$

may be expanded in a series of a peculiar form, and have thus rendered the determination of these attractions a matter of comparative facility. The same method applied to the equation (2.) of the present paper, has the advantage of giving an expansion of its general integral, every term of which, besides satisfying this equation, may likewise be made to satisfy the condition (6.). The formula (3.) is only an individual term of the expansion in question. But in order to render the present communication independent of every other, it was thought advisable to introduce into the text a demonstration of this particular case.

To prove that the expression (3.) satisfies the equation (2.), it may be remarked, that we readily get, by differentiating (3.)

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=\frac{2\mu}{a^3bc}\frac{df}{dx}+\frac{\mu x}{a^3bc}\left(\frac{d^2f}{dx^2}+\frac{d^2f}{dy^2}+\frac{d^2f}{dz^2}\right)-$$
$$-\frac{\mu x}{a^3bc}\left(\frac{3}{2a^2}+\frac{1}{2b^2}+\frac{1}{2c^2}\right)\left\{\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2+\left(\frac{df}{dz}\right)^2\right\}$$

Moreover, by the same means, the last of the equations (4.) gives

$$\frac{df}{dx}=\frac{\frac{2x}{a^2}}{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}},\quad \left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2+\left(\frac{df}{dz}\right)^2=\frac{4}{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$$

and $$\frac{d^2f}{dx^2}+\frac{d^2f}{dy^2}+\frac{d^2f}{dz^2}=\frac{\frac{2}{a^2}+\frac{2}{b^2}+\frac{2}{c^2}}{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$$

which values being substituted in the second member of the preceding equation, evidently cause it to vanish, and we thus perceive that the value (3.) satisfies the partial differential equation (2.)

We will now endeavour so to determine the constant quantities λ and μ that the fluid particles may move along the surface of the ellipsoidal body of which the equation is

$$1=\frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5.)$$

But, by differentiation, there results

$$0=\frac{x\,dx}{a'^2}+\frac{y\,dy}{b'^2}+\frac{z\,dz}{c'^2}$$

and as the particles must move along the surface, it is clear that the last equation ought to subsist, when we change the elements dx, dy and dz into their corresponding velocities $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$ and $\frac{d\phi}{dz}$. Hence, at this surface,

$$0=\frac{x}{a'^2}\frac{d\phi}{dx}+\frac{y}{b'^2}\frac{d\phi}{dy}+\frac{z}{c'^2}\frac{d\phi}{dz} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6.)$$

But the expression (3.) gives generally

$$\frac{d\phi}{dx} = \lambda + \mu\int_{\infty}\frac{df}{a^3bc} + \frac{\mu x}{a^3bc}\frac{df}{dx}, \quad \frac{d\phi}{dy} = \frac{\mu y}{a^3bc}\frac{df}{dy}, \quad \frac{d\phi}{dz} = \frac{\mu z}{a^3bc}\frac{df}{dz} \quad \text{.........} \quad (7.)$$

and, consequently, at the surface in question, where $f = 0$,

$$\frac{d\phi}{dx} = \lambda + \mu\int_{\infty}^{0}\frac{df}{a^3bc} + \frac{\mu x}{a'^3b'c'}\frac{df}{dx}, \quad \frac{d\phi}{dy} = \frac{\mu y}{a'^3b'c'}\frac{df}{dy}, \quad \frac{d\phi}{dz} = \frac{\mu z}{a'^3b'c'}\frac{df}{dz}.$$

These values, substituted in (6.), give, when we replace $\frac{df}{dx}$, $\frac{df}{dy}$, and $\frac{df}{dz}$ with their values at the ellipsoidal surface,

$$0 = \lambda + \mu\int_{\infty}^{0}\frac{df}{a^3bc} + \frac{2\mu}{a'b'c'} \quad \text{..} \quad (8.)$$

which may always be satisfied by a proper determination of one of the constants λ and μ, the other remaining entirely arbitrary.

From what precedes, it is clear, that the equation (2.), and condition to which the fluid is subject, may equally well be satisfied by making

$$\phi = \left(\lambda' + \mu'\int_{\infty}\frac{df}{ab^3c}\right) y \text{ and } \phi = \left(\lambda'' + \mu''\int_{\infty}\frac{df}{abc^3}\right) z;$$

provided we determine the constant quantities therein contained by means of the equations

$$0 = \lambda' + \mu'\int_{\infty}^{0}\frac{df}{ab^3c} + \frac{2\mu'}{a'b'c'} \text{ and } 0 = \lambda'' + \mu''\int_{\infty}^{0}\frac{df}{abc^3} + \frac{2\mu''}{a'b'c'}$$

respectively. The same may likewise be said of the sum of the three values of ϕ before given. However, in what follows, we shall consider the value (3.) only, since, from the results thus obtained, similar ones relative to the cases just enumerated may be found without the least difficulty.

Instead now of supposing the solid at rest, let every part of the whole system be animated with an additional common velocity $-\lambda$ in the direction of the co-ordinate x. Then, it is clear, that the equation (2.), and condition to which the fluid is subject, will still remain satisfied. Moreover, if x', y', z' are now referred to three axes fixed in space, we shall have

$$x' = x - \int\lambda dt, \qquad y = y', \qquad z = z'$$

and if X′ represents the co-ordinate of the centre of the ellipsoid referred to the fixed origin, we shall have

$$X' = -\int \lambda dt \quad \text{(9.)}$$

Adding now to ϕ the term $-\lambda x$ due to the additional velocity, the expression (3.) will then become

$$\phi = \mu x \int_{\infty} \frac{df}{a^3 bc}$$

and the velocities of any point of the fluid will be given, by means of the differentials of this last function. But ϕ and its differentials evidently vanish at an infinite distance from the solid, where $f = \infty$; and consequently, the case now under consideration is that of an indefinitely extended fluid, of which the exterior limits are at rest, whilst the parts in the vicinity of the moving body are agitated by its motions.

It will now be requisite to determine the pressure p at any point of the fluid mass. But, by supposing this mass free from all extraneous action $V = 0$, and if the excursions of the solid are always exceedingly small, compared with its dimensions, the last term of the second member of the equation (1.) may evidently be neglected, and thus we shall have, without sensible error,

$$-\frac{p}{\rho} = \frac{d\phi}{dt} \qquad i.e. \qquad p = -\rho \frac{d\phi}{dt}$$

or, by substitution from the last value of ϕ,

$$p = -\frac{d\mu}{dt} \rho x \int_{\infty} \frac{df}{a^3 bc}$$

Having thus ascertained all the circumstances of the fluid's motion, let us now calculate its total action upon the moving solid. Then, the pressure upon any point on its surface will be had by making $f = 0$ in the last expression, and is

$$p_0 = -\frac{d\mu}{dt} \rho x \int^{0} \frac{df}{a^3 bc}.$$

Hence we readily get for the total pressure on the body tending to increase, x

$$\mathrm{P} = \int ds\,(p_0' - p_0'') = \frac{d\mu}{dt}\,\rho \int_\infty \frac{df}{a^3bc} \times \int 2x\,ds = \frac{d\mu}{dt}\,\rho\, v \int_\infty^0 \frac{df}{a^3bc}\,;$$

v representing the volume of the body, p_0'' the pressure on that side where x is positive, p_0' the pressure on the opposite side, and ds an element of the principal section of the ellipsoid perpendicular to the axis of x.

If now we substitute for μ its value given from (8.), the last expression will become

$$\mathrm{P} = \frac{a'b'c'\,\rho\, v \int_0^\infty \frac{df}{a^3bc}}{2 - a'b'c' \int_0^\infty \frac{df}{a^3bc}}, \; \frac{d\lambda}{dt} \quad \text{..............................} \quad (10.)$$

Having thus the total pressure exerted upon the moving body by the surrounding medium, it will be easy thence to determine the law of its vibrations when acted upon by an exterior force proportional to the distance of its centre from the point of repose. In fact, let $\rho_{,}$ be the density of the body, and, consequently, $\rho_{,}v$ its mass, $g\mathrm{X}'$ the exterior force tending to decrease X'. Then, by the principles of dynamics,

$$0 = \rho_{,} v \frac{d^2\mathrm{X}'}{d^2 t} + g\mathrm{X}' - \mathrm{P}.$$

If, now, in the formula (10.) we substitute for λ its value drawn from (9.), the last equation will become

$$0 = \left(\rho_{,} + \frac{a'b'c' \int_0^\infty \frac{df}{a^3bc}}{2 - a'b'c' \int_0^\infty \frac{df}{a^3bc}}\,\rho \right) v\, \frac{d^2\mathrm{X}'}{dt^2} + g\mathrm{X}'$$

which is evidently the same as would be obtained by supposing the vibrations to take place *in vacuo*, under the influence of the given exterior force, provided the density of the vibrating body were increased from

$$\rho_{,} \text{ to } \rho_{,} + \frac{a'b'c'\int_0^\infty \frac{df}{a^3bc}}{2 - a'b'c'\int_0^\infty \frac{df}{a^3bc}}\,\rho \quad \text{.................................} \quad (11.)$$

We thus perceive, that, besides the retardation caused by the loss of weight which the vibrating body sustains in a fluid, there is a farther retardation due to the action of the fluid itself; and this last is precisely the same as would be produced by augmenting the density of the body in the proportion just assigned, the moving force remaining unaltered.

When the body is spherical, we have $a' = b' = c'$, and the proportion immediately preceding becomes very simple, for it will then only be requisite to increase $\rho_{,}$ the density of the body by $\frac{\rho}{2}$, or half the density of the fluid, in order to have the correction in question.

The next case in point of simplicity is where $a' = c'$, for then

$$\int_0^\infty \frac{df}{a^3bc} = \int_0^\infty \frac{df}{a^4b} = 2\int_{b'}^\infty \frac{db}{a^4} \quad \text{.......................................} \quad (12.)$$

If $a' > b'$, or the body, is an oblate spheroid vibrating in its equatorial plane, the last quantity properly depends on the circular arcs, and has for value

$$(a'^2 - b'^2)^{-\frac{3}{2}} \left\{ \frac{\pi}{2} - \text{arc}\left(\tan = \frac{b'}{\surd(a'^2 - b'^2)}\right) \right\} - \frac{b'}{a'^2(a'^2 - b'^2)}$$

If, on the contrary, $a' < b'$, or the spheroid, is oblong, the value of the same integral is

$$\tfrac{1}{2}(b'^2 - a'^2)^{-\frac{3}{2}} \log \frac{b' + \surd(b'^2 - a'^2)}{b' - \surd(b'^2 - a'^2)} + \frac{b'}{a'^2(b'^2 - a'^2)}$$

Another very simple case is where $c' = b'$, for then the first of the quantities (12.) becomes, if $a' > b'$

$$(a'^2 - b'^2)^{-\frac{3}{2}} \log \frac{a' + \surd(a'^2 - b'^2)}{a' - \surd(a'^2 - b'^2)} - \frac{2}{a'(a'^2 - b'^2)}$$

and if $a' > b'$, the same quantity becomes

$$2(b'^2 - a'^2)^{-\frac{3}{2}} \left\{ \text{arc}\left(\tan = \frac{a'}{\surd(b'^2 - a'^2)}\right) - \frac{\pi}{2} \right\} + \frac{2}{a'(b'^2 - a'^2)}$$

By employing the first of the four expressions immediately preceding, we readily perceive, that, when an oblate spheroid vibrates in its equatorial plane, the correction now under consideration will be affected by conceiving the density of the body augmented from

$$\rho_{,} \text{ to } \rho_{,} + \frac{\frac{\pi}{2}a'^2 b' - a'^2 b' \text{ arc}\left(\tan = \frac{b'}{\sqrt{(a'^2 - b'^2)}}\right) - b'^2 \sqrt{(a'^2 - b'^2)}}{2(a'^2 - b'^2)^{\frac{3}{2}} - \frac{\pi}{2}a'^2 b' + a'^2 b' \text{ arc}\left(\tan = \frac{b'}{\sqrt{(a'^2 - b'^2)}}\right) + b'^2 \sqrt{a'^2 - b'^2}} \rho$$

When b' is very small compared with a', or the spheroid is very flat, we must augment the density

$$\text{from } \rho_{,} \text{ to } \rho_{,} + \frac{\pi}{4}\frac{b'}{a'}\rho \text{ nearly};$$

and we thus see that the correction in question becomes less in proportion as the spheroid is more oblate.

In what precedes, the excursions of the body of the pendulum are supposed very small, compared with its dimensions. For, if this were not the case, the term of the second degree in the equation (1.) would no longer be negligible, and therefore the foregoing results might thus cease to be correct. Indeed, were we to attend to the term just mentioned, no advantage would even then be obtained; for the actual motion of the fluid, where the vibrations are large, will differ greatly from what would be assigned by the preceding method, although this method consists in satisfying all the equations of the fluid's motion, and likewise the particular conditions to which it is subject. It would be encroaching too much upon the Society's time to enter on the present occasion into an explanation of the cause of this apparent anomaly: it will be sufficient here to have made the remark, and, at the same time, to observe, that when the extent of the vibrations is very small, as we have all along supposed, the preceding theory will give the proper correction to be applied to bodies vibrating in air, or other elastic fluid, since the error to which this theory leads cannot bear a much greater proportion to the correction before assigned, than the pendulum's greatest velocity does to that of sound.