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ORIGINAL PAPER

A Brief History of the Most Remarkable Numbers π, g and δ in Mathematical Sciences with Applications

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Abstract This paper deals with a brief history of the most remarkable numbers π, g and δ in mathematical sciences with their many examples of applications. Several series, products, continued fractions and integral representations of π are discussed with examples. The celebrated Newton method of approximation of π to many decimal places is included. The appearance of π in many problems, formulas, elliptic integrals and in probability and statistics is presented with examples of applications including the Tchebycheff problem of prime numbers, the Buffon needle problem and the Euler quadratic polynomial. The golden number g and its applications to algebra and geometry are briefly discussed. The Feigenbaum universal constant, δ is discovered in 1978 and it is found to occur in many period doubling bifurcation phenomena in the celebrated logistic map and the Lorenz differential equation system with chaotic (or aperiodic) solutions. Included is a numerically computed Lorenz attractor which resembles a *butterfly* or *figure eight*. The Lorenz attractor is a *strange attractor* because it has a non-integer (or fractal) dimension. The major focus of this article is to provide basic pedagogical information through historical approach to mathematics teaching and learning of the fundamental knowledge and skills required for students and teachers at all levels so that they can understand the concepts of mathematics, and mathematics education in science and technology and pursue further research.

Keywords Universal constant π · Golden number g · Feigenbaum's constant · δ · Chaos

Mathematics Subject Classification 01A · 47A07 · 26D15

"Mathematics, rightly viewed, possesses not only truth, but supreme beauty–a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only greatest art can show."

Bertrand Russell

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"The simple is the seal of the true. And beauty is the splendor of truth."

S. Chandrasekhar

"The greatest mathematicians like Archimedes, Newton, and Gauss have always been able to combine theory and applications into one."

Felex Klein

Introduction

The mathematical numbers π, g and δ are considered as the most fundamental and useful constant numbers in the mathematical sciences. The purpose of this article is to give a short historical development of these constants with numerous classical and modern examples of applications for the students, teachers and the general readers of mathematics, science and technology. Included are some basic analytical and computational ideas that arose along the way the stories of some remarkable mathematics from some remarkable mathematicians. The main focus of the article is to use a historical approach to mathematics teaching, learning and the practical skills that students and teachers need to gain a real understanding of these constants and their applications.

Classical and Modern Origins of Archimedes's Number π

Historically, the number π and its analytical and computational aspects had received a great deal of attention to the study of geometry by Babylonians, Chinese, Egyptians, Greek and Indian mathematicians in ancient times. About 4000 years ago, the number $\pi = 3.14159...$ of antiquity was linked to the perimeter and the area of a circle by ancient geometers. More precisely, π is defined by the formula for the perimeter $\mathrm{P} = 2\pi\ r\ \left(or\ \pi = \frac{P}{2r}\right)$ of a circle of radius r, or by the formula for the area $\mathrm{A} = \pi r^2 \left(or\ \pi = \frac{A}{r^2}\right)$ of the circle of radius r. This definition implies that π is a special quantity because its value is the same for every circle, whatever its size. The ancient mathematicians paid a great deal of attention to the mathematical computation of π. In the *Ahmose (Rhind)* Papyrus, an Egyptian mathematics document dated back at least 1650 BC contained the *one – ninth computational rule* as follows. The area of a circle, $\pi\left(\frac{d}{2}\right)^2$ is equal to that of a square of side $\left(d - \frac{d}{9}\right) = \frac{8d}{9}$, that is ,$\frac{\pi d^2}{4} = \left(\frac{8d}{9}\right)^2$so that

$$\pi = \left(\frac{16}{9}\right)^2 = \frac{256}{81} \backsim 3.1604.... \tag{1}$$

Thus 3.1604... is a very fair approximation of the number π. According to several ancient stories, it was believed that the perimeter of the circular ocean was approximately equal to three times its diameter so that π would be approximately equal to 3. This fact was in full agreement with the ancient Babylonians's findings. Indeed, all backed clay tablets lifted in Mesopotamia also confirmed that in ancient Babylonians took $\pi \backsim 3$, or more accurately, $\pi = 3\frac{1}{8} = 3.125$ which was a little better approximation of π.

According to a remarkable historical fact in the Indian Hindu mathematics, Aryabhatta (476–550 BC) provided an approximate value of π as follows : $(100 + 4) \times 8 = 832$ was added to 62,000 which was used as the measure of the perimeter of a circle of diameter 20,000 so that

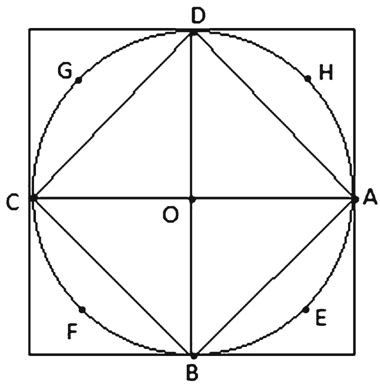

Fig. 1 A circle with an inscribed square or regular octagon

$$\pi = \frac{perimeter}{diameter} = \frac{62{,}832}{20{,}000} = 3.1416. \tag{2}$$

Amazingly, this seemed to be a very close and accurate estimate of π to four decimal places. It follows from the above discussion that there were two famous problems posed by ancient geometers. One was the length of the perimeter of a circle and the other was the area of a circle. One natural approach to these problems is to construct inside a circle square or polygon which follows the line of the circle around closely. The length of the perimeter of the polygon is the sum of the lengths of its sides. It is then natural to expect that the length of the perimeter of the polygon will be approximately equal to the length of the perimeter of the circle.

In Cartesian coordinate geometry, the standard equation of a circle of radius a with center at (0,0) is $x^2 + y^2 = a^2$. We then inscribe in it a square ABCD as shown in the Fig. 1.

The point A is (a,0) and the point D is (0,a) so that the length AD is $a\sqrt{2}$. Thus, the length of the perimeter of the square is $4a\sqrt{2}$ which is less than that of the circle. We next bisect each arc AB, BC, CD, DA by points E, F, G, H, and these give AE, EB, BF, FC, CG, GD, DH, and HA so that we obtain a regular octagon which is much closer to the circle than the square. Similarly, the length of the octagon does not have a simple expression as the length of the square. However, the length of the perimeter of the octagon can be determined.

Proceeding similarly, it is possible to insert new points on the circle midway between the old ones. This leads us to obtain regular inscribed polygon with $2^4, 2^5, 2^6, \ldots, 2^n$ sides with the perimeters $l_4, l_5, \ldots, l_n$ which form a sequence with $l_n < l_{n+1}$, where n = 2, 3, …. The sequence l_n converges to the length of the circumference, $2\pi a$ of the circle of radius a. In other words,

$$\lim_{n\to\infty} l_n = 2\pi a. \tag{3}$$

It follows from the values of the perimeters of the two squares (one inside and one outside the circle of radius a) that $2\pi a$ lies between $4a\sqrt{2}$ and $8a$, that is, $4\sqrt{2}a < 2\pi a < 8a$, or, $2\sqrt{2} < \pi < 4$.

Besides the perimeter of a circle of definite length $2\pi a$, it encloses a region with a definite area. We again use the regular polygon with 2^n sides inscribed in the circle to form 2^n triangles by joining each vertex of the polygon to the center of the circle by a straight line. Thus, the area of each triangle is equal to $\frac{1}{2}\times$ base $\times$ altitiude, where the base is equal to the side of the polygon. Then the sum of all the bases is the perimeter of the polygon, and hence, it is approximately equal to $2\pi a$ as $n \to \infty$. The altitude of each triangle is equal to the radius a so that the total area enclosed by the polygon is $\frac{1}{2} \times 2\pi a \times a = \pi a^2$ as n is very large, and consequently, the area enclosed by the circle of radius a is πa^2.

There were three famous problems which were proposed by ancient geometers, but they were never solved. The first one was that of squaring the circle. The second was that of trisecting a given angle, that is, dividing a given angle into three equal parts. The third was that of duplicating a cube, that is, to construct a cube which has twice the volume of a given cube. All these constructions, of course, have to be done by Euclidean methods. The ancient mathematicians were unable to solve these problems, not because they were not smart enough, but because the problems themselves were unsolvable at that time. The main reason was that the solutions involved irrational numbers which cannot be constructed by Euclidean methods, that is, with ruler and compasses only. We may take the radius one of a circle so that its area is π which must be equal to area of a square of side a, that is $a^2 = \pi$. Thus, the proposed square has sides equal to $\sqrt{\pi}$. since π is irrational, neither π nor $\sqrt{\pi}$ can be constructed by Euclidean methods. Hence, it is impossible to square the circle. This problem is still one of the major unsolved problems in mathematics.

In his great 225 BC treatise *The Measurement of the circle,* it was Archimedes (287–212 BC), the greatest Greek mathematical philosopher and scientist, who generalized the above link between a circle and π to calculate areas of many planar regions such as ellipse or Archimedes's spiral, and surface areas and volumes of many three-dimensional geometrical figures including sphere, cylinder, cone, and torus. All of his remarkable works were considered as celebrated discoveries of Archimedes who is regarded as one of the greatest mathematical scientists of all time, and certainly the greatest in antiquity. He was considered as most creative and original mathematical scientists of all ages as he provided unique insights into the ancient Greek mathematics, and at the same time, he made many major discoveries in mathematics, mechanics and mathematical physics. He first inaugurated the classical method of scientific computation of the number π with sufficient accuracy. He used the semiperimeter of a circle of radius one by the semiperimeters a_n and b_n of the circumscribed and the inscribed regular polygons of $6 \times 2^0 = 6, 6 \times 2^1 = 12, 6 \times 2^2 = 24, 6 \times 2^3 = 48$ and $6 \times 2^4 = 96$ sides and obtained the upper and lower bounds as

$$3.140845 = 3 + \frac{10}{71} = b_n < \pi < a_n < 3 + \frac{1}{7} = 3.1428571. \tag{4}$$

Indeed, the most famous approximation to π is $\frac{22}{7}$ which was due to Archimedes. However, a still better rational approximation to π is $\frac{355}{113}$ which was due to a Chinese mathematician, Zu Chang-Zhi (430–501 AD) as the error estimate is

$$\left|\pi - \frac{355}{113}\right| < \frac{1}{q_3 q_4} = \frac{1}{113 \times 33{,}102} < 10^{-6}, \tag{5}$$

q_3 and q_4 are partial functions of a simple but irregularly behaved continued function representation of π due to Johann Lambert (1728–1777) as

$$\begin{aligned}\pi &= 3 + \frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{292+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{14+\cdots}\\ &= [3; 7, 15, 1, 292, 1, 1, 1, 3, 1, 14, \cdots],\end{aligned} \tag{6}$$

with the first few partial fractions

$$\frac{p_n}{q_n} = \frac{3}{1}, \frac{22}{7}, \frac{333}{113}, \frac{103{,}993}{33{,}102}, \cdots \tag{7}$$

All these mean that $\frac{22}{7}$ is *is not the best approximation* to π, but $\frac{355}{113}$ is as per the error estimate (5). This fraction $\frac{355}{113} = 3.141\,59\,29$ was rediscovered by Adrian Metius (1571–1635) in 1589.

In 1593, a French lawyer, Francois Viète (1540–1603) first gave an exact, analytical infinite product formula for $\left(\frac{2}{\pi}\right)$ in the form

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} a_n, \tag{8}$$

where

$$a_{n+1} = \sqrt{\frac{1}{2}\left(1 + \sqrt{a_n}\right)} \quad \text{with} \quad a_0 = 0. \tag{9}$$

In fact, (8) becomes

$$\frac{2}{\pi} = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\right)\cdots. \tag{10}$$

Although this looks little complicated since the square roots are involved, but the numerical calculation of π to 14 decimal places is

$$\pi = 3.141\,592\,653\,589\,79.... \tag{11}$$

In 1671, Sir Isaac Newton (1642–1727) discovered an approximate value of π to 16 decimal places in his *Methodus Fluxionum et Serierum Infinitorum.* Amazingly, he began with the equation of an upper half semicircle with the center at $\left(\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$ in the form $\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$, or $y = \sqrt{x(1-x)}$ as shown in Fig. 2.

Using the binomial expansion of $\sqrt{1-x}$, the equation of the semicircle has the series form

$$\begin{aligned} y = \sqrt{x}\sqrt{1-x} &= \sqrt{x}\left(1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \frac{7}{256}x^5 - \cdots\right) \\ &= x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}} - \frac{5}{128}x^{\frac{9}{2}} - \frac{7}{256}x^{\frac{11}{2}}\cdots. \end{aligned} \tag{12}$$

Newton then used his own rule for finding areas under simple curves from his 1745 *De Analyst*. According to his fluxions rule of area of the curve $y = ax^{\frac{m}{n}}\ from\ x = 0\ to\ x = x,$

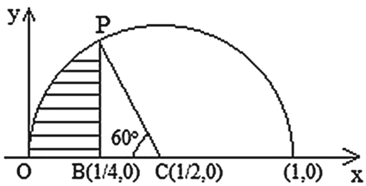

Fig. 2 A semicircle with center (1/2,0) and radius r = 1/2

the area is given by

$$A = a\frac{x^{\frac{m}{n}+1}}{\frac{m}{n}+1} = \frac{an}{m+n}x^{\frac{m+n}{n}}. \tag{13}$$

Using this formulas (12)–(13), the area of the sector OPB from $x = 0\, to\, x = \frac{1}{4}$ is given by

$$\begin{aligned}\text{A(sector OPB)} &= \frac{2}{3}\left(\frac{1}{4}\right)^{\frac{3}{2}} - \frac{1}{5}\left(\frac{1}{4}\right)^{\frac{5}{2}} - \frac{1}{28}\left(\frac{1}{4}\right)^{\frac{7}{2}} - \frac{1}{72}\left(\frac{1}{4}\right)^{\frac{9}{2}} - \frac{5}{704}\left(\frac{1}{4}\right)^{\frac{11}{2}} - \cdots \\ &= \frac{1}{12} - \frac{1}{160} - \frac{1}{3584} - \frac{1}{36{,}864} - \frac{5}{1{,}441{,}792} - \cdots \\ &\approx 0.076\,773\,106\,78\cdots. \end{aligned} \tag{14}$$

Newton also used geometrical arguments that the angle of the sector OPC is $\theta = 60^o$ which is one third of 180^o angle of the semicircle so that the area of the sector OPC is one third of the area of the semicircle minus the area of the right angled triangle $\triangle$ PBC ($\text{PC}^2 = \text{PB}^2 + \text{BC}^2$), that is,

$$\begin{aligned}\text{A(sector OPB)} &= \text{A(sector OPC)} - \text{A}(\triangle\ \text{PBC}) \\ &= \frac{1}{3}\left(\frac{1}{2}\pi r^2\right) - \frac{1}{2}\times\frac{1}{4}\times\frac{\sqrt{3}}{4} = \frac{\pi}{24} - \frac{\sqrt{3}}{32}.\end{aligned} \tag{15}$$

Equations (14) and (15) gives Newton′s approximation of π as

$$\pi = 24\left(0.076\,773\,106\,78 + \frac{\sqrt{3}}{32}\right) \approx 3.14\,159\,2656. \tag{16}$$

Although π had been calculated to about 100 decimal places by series methods, but in 1761 Lambert finally proved that π is irrational, so it cannot be written as the ratio of two natural numbers. In his 1882 classic memoir, Carl Lindemann (1852–1939) used the proof similar to that of Lambert that e is transcendental, to prove that π is also transcendental, hence, it cannot be a solution of any polynomial equation of rational coefficients. It follows from the decimal expansion of π which never terminates. Modern computers have now allowed to determine π to several billion decimal places with what objective it is difficult to say. It is surprising that there is no definite pattern in the decimal digits of π. In 1929, A. O. Gelfond (1906–1968) proved that e^{π} and $2^{\sqrt{2}}$ are transcendentals, but nothing is known about the nature of $\pi + e$, πe, π^e, π^π, e^e. However, he proved a general theorem that α^β is transcendental if α and β are algebraic numbers with $\alpha \neq 0$ and $\alpha \neq 1$ and β is *not* a real rational number.

Historically, the approximate value $\frac{22}{7}$ of π was found to appear in almost all elementary textbooks in several centuries. Later Archimedes (see Debnath [1]) first discovered the link of π in the perimeter and area of a circle of arbitrary radius, and then generalized this link to measure the area $\text{A} = \pi ab$ of an ellipse of semi−axes a and b $(< a)$ so that $\text{A} = \pi ab$ reduces to $\text{A} = \pi r^2$ when $a = b = r$. In his treatise, he also calculated the area of the Archimedes spiral as $\text{A} = \frac{1}{3}\pi(2\pi r)^2 = \frac{4}{3}\pi^3 r^2$. In addition, he also considered several three dimensional solids of revolution including sphere, ellipsoid and paraboloid and obtained the formulas for their surface areas and volumes by a method of cutting into cylindrical slices. For example, the surface area S and volume V of a sphere of radius r are $\text{S} = 4\pi r^2$ and $\text{V} = \frac{4}{3}\pi r^3$.

Around 400 BC, a famous Greek mathematician Democritus (460–370 BC) discovered the volume of a pyramid (or of a cone) is equal to one-third of the volumes of a prism (or of a cylinder) of the same base and same height. For example, the volume of a cone of height h with a circular base of radius r is $\frac{1}{3}\pi r^2 h$. The volume of a cylinder of height h with a circular base of radius r is $\pi r^2 h$. Later on, a famous Greek mathematician, Eudoxus (408–355 BC) gave a rigorous proof of these results and he became very famous for his two major contributions including the theory of proportion, and the method of exhaustion which was very useful in finding areas and volumes of many geometrical figures. He is usually considered as the finest mathematician of antiquity next to Archimedes himself.

Using the Archimedes method, the volume of a paraboloid inscribed in a cylinder of radius r and height h is $\frac{1}{2}\pi r^2 h$ which is the half of the volume of the ellipsoid generated by rotating the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ around the y-axis is equal to $\frac{4}{3}\pi a^2 b$. On the other hand, Pappus of Alexandria (290–350 AD) found the volume V $= 2\pi^2 r a^2$ generated by rotating the circle $(x-r)^2 + y^2 \leq a^2$ with $r > a$. about the y-axis. In fact, the volume V is the product of the area of the meridional circle and the distance moved by its center.

It is important to point out that all of the above results are associated with π. So, the n-dimensional volume in the Euclidean space $\mathbb{R}^n$ of a hypersphere of radius r is

$$V_n(r) = \underset{x_1^2+x_2^2+\cdots+x_n^2 \leq r^2}{\int\int\int \cdots \int} dx_1 dx_2 \cdots dx_n. \tag{17}$$

$$= \frac{2r^n}{n} \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}, \tag{18}$$

where $\Gamma(x)$ is the Euler gamma function which will be defined later on by (95). In particular, when $n = 2$, we have the area of a circle $= \pi r^2$, when $n = 3$, we have the volume of a sphere V $= \frac{4}{3}\pi r^3$, and so on.

Differentiating V_n(r) with respect to r gives the surface area S of a hypersphere

$$S = \frac{2r^{n-1}\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \tag{19}$$

Thus, for $n = 2$, S $= 2\pi r$, the circumference of a circle, for $n = 3$, S = $4\pi r^2 = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)$ is the surface area of a sphere, and so on. The above discussion reveals that π appears in many formulas in geometry.

Similarly, the volume of a hyperellipsoid of equation

$$\frac{x_1^2}{r_1^2} + \frac{x_2^2}{r_2^2} + \cdots + \frac{x_n^2}{r_n^2} = 1 \tag{20}$$

is given by

$$V_n = \frac{2r_1 r_2 \cdots r_n}{n} \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \tag{21}$$

Thus, for $n = 2$, S $= \pi a_1 a_2 = \pi ab$ is the perimeter of an ellipse, $n = 3$ with $r_1 = r_2 = r_3 = r$, the volume $\mathrm{V}_3 = \frac{4}{3}\pi r^3$, the volume of a sphere, and so on.

Carl Friedrich Gauss (1777–1855), a great German mathematician formulated a celebrated *circle problem* which linked π with the arithmetic mean of the number of points on the circle $x^2 + y^2 = n$. He considered a lattice G consisting of a set of points (x, y) with integer coordinates on the above circle. With $r(n)$ as the number of decomposition of n as a sum of two squares, Gauss observed that $r(n)$ has the chaotic behaviour as $n \to \infty$, and hence, examined the behaviour of the arithmetic mean so that

$$\lim_{n\to\infty} \frac{1}{n}\left[r(1)+r(2)+\cdots+r(n)\right]=\pi \tag{22}$$

Or, equivalently, (22) means that

$$r(1)+r(2)+\cdots+r(n)=n\pi + O(\sqrt{n}). \tag{23}$$

This has a clear interpretation that the left hand side of (23) is approximately equal to the area of the circle of radius $\sqrt{n}$ with an error less than the perimeter of the circle.

Thus, the Gauss circle problem involving π received considerable attention of Carl Jacob Jacobi (1804–1851) and Peter Gustav Dirichlet (1805–1859), and hence, it became a major subject of research in the twentieth century. In particular the error term $O(n^{\varepsilon})$ became the major focus of many mathematicians with a conjecture that the lower bound $\varepsilon_{\circ} = \frac{1}{4}$ of ε. In 1834, Gauss proved $\varepsilon_{\circ} \le \frac{1}{2}$. In 1915, a great British pure mathematician, G.H. Hardy (1877–1947) and a celebrated Russian mathematical scientist, Edmund Landau (1877–1938) showed that $\varepsilon_{\circ} \ge \frac{1}{4}$. In 1906, a great Polish mathematician, Waclaw Sierpinski (1882–1969) proved that $\varepsilon\circ \le \frac{1}{3}$. After a series of improvements of the lower bound of ε, the best upper bound $\varepsilon \le \frac{22}{72}$ is found by Martin Huxley in 1993. Subsequently, the Gauss circle problem was generalized for the case $n = ax^2 + by^2$, where a and b are two positive integers so that

$$r(1)+r(2)+\cdots+r(n)=\frac{\pi n}{\sqrt{ab}} + O(\sqrt{n}). \tag{24}$$

If the circle is replaced by a branch of an equilateral hyperbolic curve that $r(n)$ can be replaced by $d(n)$ which is the number of divisors of n including 1 and n itself. Using the fundamental theorem of arithmetic for the unique prime factorization of n as

$$n = p_1^{k_1} p_2^{k_2}\cdots p_\ell^{k_\ell} = \prod_{r=1}^{\ell} p_r^{k_r}, \tag{25}$$

the function $d(n)$ takes the form

$$d(n)=\prod_{s=1}^{\ell}(k_s+1). \tag{26}$$

The function $d(n)$ is multiplicative , that is, $d(nm) = d(m)\, d(n)$ if m and n are coprimes as $n \to \infty$, $d(n)$ has a chaotic behaviour. So, we take the arithmetic mean of $d(n)$, and find that, as $n \to \infty$

$$\frac{d(1)+d(2)+\cdots+d(n)}{n} \backsim ln\ n \tag{27}$$

According to Dirichlet's estimate of 1849 as $n\to\infty$,

$$d(1)+d(2)+\cdots+d(n) = n\ ln\ n + (2\gamma - 1)\, n + O(\sqrt{n}), \tag{28}$$

where $\gamma = 0.57721$ is the celebrated Euler constant.

The whole approach changed completely with the advent of calculus in the mid-seventeenth century. Many striking formulas for π were discovered in trigonometry and calculus. In 1656, John Wallis (1616–1703), a famous British mathematician and a founding member of the Royal Society of London in 1660, first discovered another simple infinite product formula for π:

$$\frac{\pi}{2} = \left(\frac{2}{1} \times \frac{2}{3}\right)\left(\frac{4}{3} \times \frac{4}{5}\right)\left(\frac{6}{5} \times \frac{6}{7}\right) \cdots = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n)^2 - 1}. \tag{29}$$

This formula was the first in which π was represented as a limit of a sequence of rational numbers. This formula (29) is somewhat simpler than Viēte's result (10) as it contains no square roots.

In 1670, another infinite series formula for π was discovered by James Gregory (1638–1675), a Scotish mathematician in the form

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^n \frac{1}{2n+1} + \cdots. \tag{30}$$

A little later, in 1674, a renowned German mathematician, Gottfried Wilhelm Leibniz (1646–1716) published his infinite series formula for π which is the same as (30). Both Gregory's and Leibniz's results are special case of the famous power series for $\tan^{-1} x$ discovered by Gregory for $-1 < x < 1$ as follows:

$$\begin{aligned} \tan^{-1} x &= \int_0^x \frac{dt}{1+t^2} = \int_0^x \left(1 - t^2 + t^4 - t^6 + \cdots\right) dt \\ &= c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \infty. \end{aligned} \tag{31}$$

When $x = 0$, $\mathrm{c} = \tan^{-1} 0 = 0$, hence, it turns out that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}. \tag{32}$$

Putting $x = 1$ in (32) gives the beautiful Gregory or Leibniz formula (30) for $\pi = 4 \tan^{-1} 1$. Although the series (30) is known as the *Gregory-Leibniz series* for $\frac{\pi}{4}$, but it was originally discovered by the Indian mathematician Nilakantha (1445–1545) much earlier.

The Wallis product formula (29) is equivalent to

$$\begin{aligned} \frac{2}{\pi} &= \left(\frac{1}{2} \times \frac{3}{2}\right)\left(\frac{3}{4} \times \frac{5}{4}\right)\left(\frac{5}{6} \times \frac{7}{6}\right) \cdots \\ &= \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{6^2}\right) \cdots. \end{aligned} \tag{33}$$

The formula (33) is a special case of celebrated product formula for $\frac{\sin \mathrm{x}}{\mathrm{x}}$:

$$\frac{\sin\ x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right) \cdots. \tag{34}$$

This can be proved by elementary arguments as follows:

To prove (34), we begin with the trignometric equation $\sin\ x = 0$ so that the Taylor series expansion of $\sin\ x$ gives

$$0 = \sin\ x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

which is convergent for all x, Euler considered $\sin\ x$ as a polynomial of infinte degree with an infinte number of roots $0, \pm\pi, \pm 2\pi, \pm 3\pi, \cdots$, so that $sin\ x$ can be expressed as the product of factors

$$(x-0)(x^2-\pi^2)(x^2-4\pi^2)\cdots(x^2-n^2\pi^2)\cdots.$$

Thus, we have with a constant A,

$$\sin\ x = Ax\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\cdots\left(1-\frac{x^2}{n^2\pi^2}\right)\cdots$$

Since $\frac{\sin\ x}{x} \to 1$ as $x \to 0$ so that A=1. Consequently, sin x has both infinte series and infinte product representation

$$\sin\ x = x\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right) = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots. \quad (35)$$

This is one of the most elegant and remarkable discoveries of Euler who equated the coefficients of x^2 in his infinite series and infinite products in (35) to obtain another more remarkable result

$$-\frac{1}{3!} = -\frac{1}{\pi^2}-\frac{1}{2^2\pi^2}-\frac{1}{3^2\pi^2}-\cdots.$$

Thus,

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}. \quad (36)$$

This is the Euler celebrated solution of the *Basel problem* dealing with the exact value of the sum of the squares of the reciprocals of the positive integers $n \geq 1$. This formula (36) involves π^2. Many infinite series, Fourier series and infinite products involve π or π^2.

Several other Euler's discovery of the extraordinary sums of infinite series are proved below. The sum of the reciprocals of the squares of even integers is given by

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\cdots=\frac{1}{4}\left(1+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\cdots\right)=\frac{1}{4}\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{24}. \quad (37)$$

Making references to result (7.3.62) given in Debnath′s [1] book and replacing z by πz in (7.3.62) gives

$$\pi z\cot\pi z = 1+\sum_{k=1}^{\infty}\frac{(-1)^k(2\pi)^{2k}B_{2k}z^{2k}}{(2k)!}, \quad (38a)$$

where the Bernoulli's numbers B_{2k} are $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$, $B_6=\frac{1}{42}$, $B_8=\frac{1}{30}$ and $B_{10}=\frac{5}{66}$.

Using result (8.4.3) in Debnath′s [1] book, we obtain

$$\pi z\cot\pi z = 1+\sum_{n=1}^{\infty}\frac{(-2)z^2}{(n^2-z^2)} = 1+\sum_{n=1}^{\infty}(-2)\sum_{k=1}^{\infty}\left(\frac{z^2}{n^2}\right)^k. \quad (38b)$$

Equations (38a) and (38b) yields the Euler′s famous result

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = \frac{(-1)^k(2\pi)^{2k}}{2(2k)!}B_{2k}, \quad k=1,2,3,\cdots. \quad (39)$$

When k = 1, (39) gives (36). If k = 2, 3, 4, 5 gives the sums of several beautiful numerical series

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}, \quad \sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{945}, \quad \sum_{n=1}^{\infty}\frac{1}{n^8}=\frac{\pi^8}{9550}, \quad \sum_{n=1}^{\infty}\frac{1}{n^{10}}=\frac{\pi^{10}}{93{,}555}. \quad (40)$$

Euler's and Machin's Formulas for π as a Sum of Inverse Tangent Functions

Both Euler and John Machin (1680–1752) discovered numerous formulas for $\frac{\pi}{4}$ as the sum of inverse tangent functions of smaller arguments so that the Gregory-Leibniz series (30) converges much more rapidly. In 1755, Euler generalised (32) in the form

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!} \frac{x^{2n+1}}{(1+x^2)^{n+1}} \tag{41}$$

with a special case for $x = 1$ so that

$$\frac{\pi}{2} = 2\tan^{-1} 1 = \sum_{n=0}^{\infty} \frac{2^n (n!)^2}{(2n+1)!}. \tag{42}$$

We prove (41) as follows :

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2}.$$

Putting $t = x\sqrt{1-s}$ so that $dt = \frac{x\,ds}{2\sqrt{1-s}}$ and $1+t^2 = 1+x^2(1-s) = (1+x^2)\left(1 - \frac{x^2 s}{1+x^2}\right)$, we obtain

$$\begin{aligned}
\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} &= \frac{x}{1+x^2} \int_0^1 \frac{1}{\left(1 - \frac{x^2 s}{1+x^2}\right)} \frac{ds}{2\sqrt{1-s}} \\
&= \left(\frac{x}{1+x^2}\right) \int_0^1 \sum_{n=0}^{\infty} \left(\frac{x^2 s}{1+x^2}\right)^2 \frac{ds}{2\sqrt{1-s}}, \quad \text{by GP series} \\
&= \left(\frac{x}{1+x^2}\right) \sum_{n=0}^{\infty} a_n \left(\frac{x^2}{1+x^2}\right)^n,
\end{aligned}$$

where $a_n = \int_0^1 \frac{s^n ds}{2\sqrt{1-s}} = \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \, d\theta = \frac{2^{2n}(n!)^2}{(2n+1)!}$.

Consequently,

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!} \times \frac{x^{2n+1}}{(1+x^2)^{n+1}}.$$

Euler also discovered numerous formuals for $\frac{\pi}{4}$ and we state a few examples as follows:

$$\begin{aligned}
\frac{\pi}{4} &= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}, &(43)\\
&= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}, &(44)\\
&= 3\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{2}{11}, &(45)\\
&= 5\tan^{-1}\frac{1}{7} + 2\tan^{-1}\left(\frac{3}{79}\right). &(46)
\end{aligned}$$

In general,

$$\frac{\pi}{4} = a_1 \tan^{-1} x_1 + a_2 \tan^{-1} x_2 + \cdots + a_n \tan^{-1} x_n, \tag{47}$$

where $x_1, x_2, \cdots, x_n$ are rational numbers and $a_1, a_2, \cdots, a_n$ integers.

We include Euler's proofs of (43)–(46), using the standard identity,

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}, \tag{48}$$

and then putting $\tan\alpha=\frac{a}{b}$ and $\tan\beta=\frac{c}{d}$ so that $\alpha=\tan^{-1}\left(\frac{a}{b}\right)$ and $\beta=\tan^{-1}\left(\frac{c}{d}\right)$.

Consequently, (48) gives

$$\tan^{-1}\left(\frac{a}{b}\right)-\tan^{-1}\left(\frac{c}{d}\right)=\tan^{-1}\left(\frac{ad-bc}{ac-bd}\right). \tag{49}$$

Putting $a=b=c=1\, and\, d=2$ in (49) gives (43). Similarly, putting $a=1, b=2, c=1\, and\, d=7$ and using (43) yields (44). Substituting $a=2, b=11, c=1\, and\, d=7$ combined with (44) leads to (45).

Euler also proved a beautiful integral formula involving π as

$$\frac{\pi}{4}=\int_0^1\frac{\sin(\ln x)}{\ln x}\,dx=\mathrm{I}\ (\text{say}). \tag{50}$$

He used the power series formula for *sin x and* then replaced the integral of the series by the series of the integrals to obtain

$$\mathrm{I}=\int_0^1\frac{\sin(\ln x)}{\ln x}\,dx=\int_0^1 dx-\frac{1}{3!}\int_0^1(\ln x)^2dx+\frac{1}{5!}\int_0^1(\ln x)^4dx-\cdots, \tag{51}$$

where the integrals $\int_0^1(\ln x)^n\,\mathrm{d}x$ are evaluated using Johann Bernoulli's reduction formula based on integration by parts in the form

$$\int x^m(\ln x)^n\,dx=\frac{1}{m+1}\,x^{m+1}(\ln x)^n-\frac{n}{m+1}\int x^n(\ln x)^{n-1}dx. \tag{52}$$

It is easy to check that

$$\int_0^1(\ln x)^2dx=\left[x(\ln x)^2-2x\ln x+2x\right]_0^1=2=2!,$$
$$\int_0^1(\ln x)^4dx=\left[x(\ln x)^4-2x(\ln x)^3+12x(\ln x)^2-24x\ln x+24x\right]_0^1=24=4!,$$

and, in general,

$$\int_0^1(\ln x)^n dx=n!$$

Using these results, the integral in (51) reduces to

$$\begin{aligned}I&=1-\frac{2!}{3!}+\frac{4!}{5!}-\frac{6!}{7!}+\cdots\\&=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}.\end{aligned} \tag{53}$$

Finally, in 1706, John Machin proved a new formula for $\frac{\pi}{4}$ as

$$\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}. \tag{54}$$

Appearance of π in Ramanujan's Formulas and in Elliptic Integrals

Euler discovered many new ideas, results and methods of partition of numbers as a sum of integers. He presented many elementary but remarkable results in his fundamental treatise on analysis, *Introductio in analysin infinitorum.* The *partition function* $p(n)$ is defined to be the number of ways of writing a positive integer as a sum of strictly positive integers. For example, $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$, so p(4) = 5. Similarly, p(5) = 7 and p(6) = 11. It is clear that p(n) increases very rapidly with n. For n = 200, p(200) = 397, 299, 902, 938, 8. This leads to the exact or asymptotic representation of $p(n)$ for large n. During the early part of the 20th century, a celebrated British pure mathematician, G.H. Hardy (1877–1947) and Indian mathematical genius, S. Ramanujan (1887–1920) discovered an asymptotic formula for p(n) as n $\to \infty$ in 1918. They proved that

$$p(n) \backsim \frac{1}{4n\sqrt{3}}\, exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{as } n \to \infty. \tag{55}$$

This is one of the most remarkable and beautiful formulas in the theory of numbers involving π. For more information of the partition function and its abundance applications in mathematical physics, the reader is referred to Debnath [1].

Ramanujan [2] also discovered many new and remarkable formulas involving $\frac{1}{\pi}$. Two sample examples are proved by him in 1914 and given below:

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 (6n+1)}{(n!)^3 4^n}, \tag{56}$$

$$\frac{2\sqrt{3}}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3} \frac{(8n+1)}{9^n}. \tag{57}$$

Another striking feature is that there are abundance appearance of π in the theory of elliptic and associated functions (see Dutta and Debnath [3]). Included below are two sample examples of applications in elliptic integrals and elliptic functions which involve π. The first one deals with the nonlinear equation of motion of a simple pendulum of length ℓ which is given by (see Dutta and Debnath [3])

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0, \quad \omega^2 = \frac{g}{\ell}, \tag{58}$$

where θ(t) is the angular displacement function with the initial displacement θ (t = 0) = α and the initial velocity $\dot{\theta}$ (t = 0) = 0.

Integrating (58) and finding the integrating constant gives

$$\left(\frac{d\theta}{dt}\right)^2 = 2\omega^2(\cos\theta - \cos\alpha) = 4\omega^2\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right). \tag{59}$$

Integrating again and putting $\sin\frac{\theta}{2} = 4\sin\frac{\alpha}{2}$ with $k = \sin\frac{\alpha}{2}$ yields

$$2\omega t = 2\omega\int_0^t dt = \int_0^\theta \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} = \int_0^u \frac{2du}{\sqrt{(1-u^2)(1-k^2u^2)}}. \tag{60}$$

This represents the *Jacobi elliptic integral of the first kind* with the Jacobi elliptic function, $u = sn(\omega t, k)$ as the solution.

When $\theta = \alpha, u = 1$ and the periodic time $T = 4t$ for a complete oscillation of the simple pendulum using u=sin ϕ is given by

$$T = 4t = \frac{4}{\omega}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{4}{\omega}\, K\,(k), \tag{61}$$

where $K(k)$ is called the *complete elliptic integral of the first kind* defined by

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \tag{62}$$

$$= \int_0^{\frac{\pi}{2}} \left(1 + \frac{1}{2}k^2\sin^2\phi + \frac{1\cdot 3}{2\cdot 4}\,k^4\sin^4\phi + \cdots\right) d\phi$$

$$= \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots\right) = \frac{\pi}{2}\,\mathrm{F}\left(\frac{1}{2}, \frac{1}{2}, 1; k^2\right), \tag{63}$$

where $F(a, b, c; z)$ is the standard hypergeometric function obtained using the definite integral

$$\int_0^{\frac{\pi}{2}} \sin^n\phi\, d\phi = \frac{1\cdot 3\cdots(2n-3)(2n-1)}{2\cdot 4\cdots(2n-2)(2n)}\;\frac{\pi}{2}. \tag{64}$$

Although (61) is the exact solution for the periodic time T, but for small k,

$$T \backsim \frac{2\pi}{\omega}\left(1 + \frac{1}{4}k^2\right) + O(k^4).$$

In the limit as $k \to 0$, (61) gives the expected periodic time for the linearized motion of the simple pendulum

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{65}$$

The second example of application deals with the *rectification (or length)* of an arc of an ellipse whose parametric equation is $x = a\ \sin\theta$ and $y = b\cos\theta$, $a > b > 0$ and $0 \le \theta \le 2\pi$. It follows from elementary calculus that the elementary arc length ds is

$$ds^2 = dx^2 + dy^2 = (a^2\cos^2\theta + b^2\sin^2\theta)d\theta^2 = a^2(1 - k^2\sin^2\theta)d\theta^2, \tag{66}$$

where $k = \left(1 - \frac{b^2}{a^2}\right)^{\frac{1}{2}}$ is called the *eccentricity* of the ellipse and $0 < k < 1$.

The length of the arc from A(a, 0) to P(x, y) is given by

$$s = \int_0^s ds = a\int_0^\theta (1 - k^2\sin^2\theta)^{\frac{1}{2}} d\theta = a\ E(k,\theta), \tag{67}$$

where θ is the angle made by the line joining the origin and the *point* $P(x, y)$ on the ellipse, and E(k, θ) is called the *elliptic integral of the second kind* defined by

$$E(k,\theta) = \int_0^\theta (1 - k^2\sin^2\theta)^{\frac{1}{2}} d\theta \tag{68}$$

The total length of the ellipse is

$$s = 4a\ E(k, \frac{\pi}{2}) = 4a \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2\theta)^{\frac{1}{2}} d\theta \tag{69}$$

$$= 4a \int_0^{\frac{\pi}{2}} \left[1 - \frac{1}{2}k^2 \sin^2\theta - \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}k^4 \sin^4\theta - \cdots\right] d\theta, \tag{70}$$

$$= 4a \times \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1\times 3}{2\times 4}\right)k^4 - \cdots\right], \tag{71}$$

$$= 2a\pi\ F(-\frac{1}{2}, \frac{1}{2}, 1; k^2), \tag{72}$$

where integrals are evaluated by (64) and $F(a, b, c; x)$ is the standard hypergeometric function.

In the limit as $k \to 0$ $(a = b)$, the total length of the circle of radius a is the known result

$$s = 4a \times \frac{\pi}{2} = 2a\pi. \tag{73}$$

The general theory of elliptic integrals and elliptic functions is beyond the scope of this article, and is referred to the book by Dutta and Debnath [3].

Appearance of π in Probability and Statistics

The number π appears abundantly in probability and statistics in view of the fact that the factorial function, $n!$ occurs in numerous problems in probability and statistics. This function is approximated in 1730 by the celebrated British mathematician, James Stirling's (1692–1770) formula in terms of π as

$$n! \sim \sqrt{2\pi}\ n^{n+\frac{1}{2}}\ e^{-n}, \quad \text{as } n \longrightarrow \infty. \tag{74}$$

If we put $n = x$, where n is a positive integer in (74), we obtain (75). We can also make some improvement of the asymptotic result (74) to obtain, for $x \longrightarrow \infty$,

$$\Gamma(x+1) \sim \sqrt{2\pi}\ x^{x+\frac{1}{2}}\ e^{-x}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \cdots\right), \tag{75}$$

where $\Gamma(\mathrm{x})$ is the Euler gamma function defined by (80). Putting $x = n$, we get a more accurate version of Stirling's formula (74) for large n in the form

$$n! \sim \sqrt{2\pi}\ n^{n+\frac{1}{2}}\ e^{-n}\left(1 + \frac{1}{12n}\right) \quad \text{as} \quad n \gg 1. \tag{76}$$

To prove (76), we write

$$e^{u_n} = \frac{n!}{n^{n+\frac{1}{2}}\ e^{-n}}, \qquad \text{or} \qquad u_n = log\left(\frac{n!}{n^{n+\frac{1}{2}}\ e^{-n}}\right)$$

and then we obtain

$$
\begin{aligned}
u_{n+1}-u_n &= log\left[\frac{(n+1)!}{(n+1)^{n+\frac{3}{2}}\, e^{-n-1}} \times \frac{n^{n+\frac{1}{2}}\, e^{-n}}{n!}\right] = 1 + log\left(\frac{n}{n+1}\right)^{n+\frac{1}{2}}\\
&= 1-\left(n+\frac{1}{2}\right) log\left(1+\frac{1}{n}\right)\\
&= 1-\left(n+\frac{1}{2}\right)\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\cdots\right]\\
&= -\frac{1}{12n^2}+\frac{1}{12n^3}-\frac{3}{40n^4}+\frac{1}{15n^5}-\cdots\\
&\approx -\frac{1}{12}\left(\frac{1}{n^2}-\frac{1}{n^3}+\frac{1}{n^4}-\frac{1}{n^5}+\cdots\right), \text{ approximately}\\
&= -\frac{1}{12n(n+1)} = \frac{1}{12}\left(\frac{1}{n+1}-\frac{1}{n}\right)
\end{aligned}
$$

Thus, we write, with a constant A,

$$u_n = \frac{1}{12n} + A,$$

Consequently,

$$\frac{n!}{n^{n+\frac{1}{2}}\, e^{-n}} = B\, exp\left(\frac{1}{12n}\right), \tag{77}$$

where B is another constant to be determined next.

We determine B by using the general well-known product formula for the sin function

$$\sin \pi x = \pi x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\left(1-\frac{x^2}{3^2}\right)\cdots.$$

Putting $x=\frac{1}{2}$ gives

$$
\begin{aligned}
\frac{2}{\pi} &= \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\cdots\\
&= \frac{1\times 3}{2^2}\times\frac{3\times 5}{4^2}\times\frac{5\times 7}{6^2}\cdots\\
&= \frac{1^2\times 3^2\times 5^2\times 7^2\cdots}{2^2\times 4^2\times 6^2\times 8^2\cdots} = \frac{1^2\times 2^2\cdot 3^2\times 4^2\cdots}{2^4\times 4^4\times 6^4\cdots}.
\end{aligned}
$$

Thus,

$$\frac{2}{\pi} = \lim_{n\to\infty}\frac{[(2n+1)!]^2}{(2n+1)(n!)^4\, 2^{4n}}. \tag{78}$$

This can be approximated by (77), and can be shown that

$$\frac{2}{\pi} = \frac{4}{B^2} \quad \text{or} \quad B = \sqrt{2\pi}, \tag{79}$$

and so (77) gives (76) The remarkable fact is that we can calculate from (76) to obtain $1! \backsim 0.99898$, $2! \backsim 1.99896$, $3! \backsim 5.99833$, $6! \backsim 719.94$ which has a small error of only 8.3×10^{-3} percent, and so on. Thus, we conclude that (76) is accurate enough for many applications for *all* positive integers.

The result (74) follows from the integral representation of the Euler gamma function (95)

$$\Gamma(x+1) = \int_0^\infty t^x\, e^{-t}\, dt. \tag{80}$$

Substituting $t = x + s$ in this integral gives

$$\Gamma(x+1) = x^x\, e^{-x} \int_{-x}^\infty exp\left[x\, ln\left(1+\frac{s}{x}\right)\right] e^{-s} ds. \tag{81}$$

For large x and fixed s, we use the approximation

$$ln\left(1+\frac{s}{x}\right) \backsim \frac{s}{x} - \frac{s^2}{2x^2} \qquad as \quad x \longrightarrow \infty. \tag{82}$$

which reduces (81) in the form

$$\Gamma(x+1) \backsim x^x\, e^{-x} \int_{-\infty}^\infty e^{\frac{-s^2}{2x}}\, ds \tag{83}$$

which is, using $u = \frac{s}{\sqrt{2x}}$,

$$\Gamma(x+1) \backsim \sqrt{2x}\, x^x\, e^{-x} \int_{-\infty}^\infty e^{-u^2}\, du \tag{84}$$

$$= \sqrt{2\pi}\, x^{x+\frac{1}{2}}\, e^{-x}, \qquad x \longrightarrow \infty. \tag{85}$$

where the *Gaussian* probability integral in (84) can be evaluated by many different ways.

In 1730, Euler evaluated the Gauss integral in (84) in terms of π as

$$I = \int_{-\infty}^\infty e^{-u^2}\, du = 2\int_0^\infty e^{-u^2}\, du = \sqrt{\pi}. \tag{86}$$

His method of evaluation is as follows :

$$I^2 = 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)}\, du\, dv$$

which is, using $v = xu$, $dv = udx$,

$$= 4\int_0^\infty dx \int_0^\infty u\, e^{-u^2(1+x^2)}\, du = 4\int_0^\infty \left[-\frac{e^{-u^2(1+x^2)}}{2(1+x^2)}\right]_0^\infty dx$$

$$= 2\int_0^\infty \frac{dx}{1+x^2} = 2\left[\tan^{-1} x\right]_0^\infty = 2 \times \frac{\pi}{2} = \pi. \tag{87}$$

More generally, for $t > 0$, integral (86) gives

$$I(t) = \int_{-\infty}^\infty e^{-tx^2}\, dx = 2\int_0^\infty e^{-tx^2}\, dx = \sqrt{\frac{\pi}{t}}. \tag{88}$$

We evaluate the integral (88) by the Gauss ingeneous method as follows:

$$I^2(t) = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-t(x^2+y^2)}\, dxdy \;= 4\int_0^\infty \int_0^\infty e^{-t(x^2+y^2)}\, dxdy, \tag{89}$$

which is, using the transformations $x = r\cos\theta$, $y = r\sin\theta$,

$$= 4\int_0^{\pi/2} d\theta \int_0^\infty J e^{-tr^2} dr = 2\pi \int_0^\infty r e^{-tr^2}\, dr = \frac{\pi}{t}\left[-e^{-tr^2}\right]_0^\infty = \frac{\pi}{t} \tag{90}$$

where the $Jacobian\ J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & y_r \\ x_\theta & y_\theta \end{vmatrix} = r.$

Putting $t = 1$ in (88) and using successive reduction leads to more general result

$$\int_0^\infty x^n e^{-x^2}\, dx = \begin{bmatrix} \frac{(n-1)(n-3)\cdots 1}{(\sqrt{2})^n} \int_0^\infty e^{-x^2}\, dx, & n(\geq 2) \text{ even} \\ \frac{(n-1)(n-3)\cdots 1}{(\sqrt{2})^{n-1}} \int_0^\infty x e^{-x^2}\, dx, & n(\geq 3) \text{ odd} \end{bmatrix}. \tag{91}$$

It also follows that the above double integral (89) is rotationally invariant which means that it depends only on r and independent of θ. Thus, the rotation involves circles and hence, involves π. It is important to point out that the function

$$f(x) = \frac{1}{\sqrt{2\pi}}\, exp\left(-\frac{x^2}{2}\right) \tag{92}$$

is called the *standard normal (or the Gaussion) distribution* function which involves π due to reasons explained above. It occurs in a wide variety of mathematical problems from probability theory to Fourier analysis and to quantum mechanics.

We can also evaluate (88) using the Laplace transform $\bar{\text{I}}$(s) of I(t) (see Debnath and Bhatta [4]) so that

$$\begin{aligned} \overline{I}(s) &= 2\int_0^\infty \mathcal{L}\left\{e^{-tx^2}\right\} dx = 2\int_0^\infty \frac{dx}{x^2+s} \\ &= \left[\frac{2}{\sqrt{s}} \tan^{-1} \frac{x}{\sqrt{s}}\right]_0^\infty = \frac{2}{\sqrt{s}} \times \frac{\pi}{2} = \frac{\pi}{\sqrt{s}}. \end{aligned} \tag{93}$$

Thus, the inverse Laplace tansform gives

$$I(t) = \mathcal{L}^{-1}\{\overline{I}(s)\} = \mathcal{L}^{-1}\left\{\frac{\pi}{\sqrt{s}}\right\} = \pi \times \frac{1}{\sqrt{\pi t}} = \sqrt{\frac{\pi}{t}} \tag{94}$$

so that

$$I(1) = \sqrt{\pi} \quad \text{and} \quad I\left(\frac{1}{2}\right) = \sqrt{2\pi}.$$

We next use the well known definition of continuous and infinitely differentiable Euler's gamma function

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt, \qquad x > 0, \tag{95}$$

to evaluate (81) by putting $u^2 = t$, $2\,u\,du = dt$ in (86) so that

$$I = \int_{-\infty}^\infty e^{-u^2}\, du = 2\int_0^\infty e^{-u^2}\, du = \int_0^\infty e^{-t} t^{-\frac{1}{2}}\, dt = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \tag{96}$$

which is obtained from the standard results of the gamma and beta functions (see page 619 of the book by Debnath and Bhatta [4]). This book also contains many single or double infinite integrals which involve π.

Integrating the integral (80) for Γ(x+1) by parts gives the celebrated *functional equation*

$$\Gamma(x+1) = x\Gamma(x) = x(x-1)\Gamma(x-1). \tag{97}$$

In particular,

$$\Gamma\left(n+\frac{1}{2}\right)=\Gamma\left(n-\frac{1}{2}+1\right)=\left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$$
$$=\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{3}{2}\times\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$
$$=\frac{1\times 3\times 5\cdots(2n-1)}{2^n}\sqrt{\pi}=\frac{(2n)!}{2^{2n}n!}\sqrt{\pi}. \tag{98}$$

It is important to mention the Gauss generalization of his integral in (84) as

$$\int_0^\infty e^{-t^x}dt=\Gamma\left(1+\frac{1}{x}\right),\quad \mathrm{x}>0. \tag{99}$$

To evaluate (86), we use a double integral in the form

$$J(t)=\int_0^\infty\int_0^\infty x\;exp[-tx^2(y^2+1)]dxdy,\quad t>0 \tag{100}$$

which is, evaluating the x-integral first,

$$\int_0^\infty\left[\frac{-1}{2t(y^2+1)}exp\left\{-tx^2(y^2+1)\right\}\right]_0^\infty dy$$
$$=\frac{1}{2t}\int_0^\infty\frac{dy}{y^2+1}=\frac{1}{2t}\left[\tan^{-1}y\right]_0^\infty=\frac{\pi}{4t}. \tag{101}$$

Thus,

$$\frac{\pi}{4t}=J(t)=\int_0^\infty e^{-tx^2}dx\int_0^\infty xe^{-tx^2y^2}dy$$

which is, putting $xy=u$,

$$=\int_0^\infty e^{-tx^2}dx\int_0^\infty e^{-tu^2}du=\left(\int_0^\infty e^{-tx^2}dx\right)^2=I^2(t).$$

Thus,

$$I(t)=\int_0^\infty e^{-tx^2}dx=\sqrt{\frac{\pi}{4t}},\quad t>0. \tag{102}$$

Finally, we consider a new double integral to prove (86) in the form

$$K=\int_0^\infty\int_0^\infty e^{-tx^2}\,\sin t\;dt\;dx. \tag{103}$$

which is, using the Laplace transform of $\sin t$,

$$=\int_0^\infty\frac{dx}{1+x^4}=\frac{\pi}{2\sqrt{2}},\quad\text{by a standard table of definite integral.} \tag{104}$$

We next evaluate (103) by putting $x\sqrt{t}=u$ so that

$$K=\int_0^\infty\frac{\sin t}{\sqrt{t}}\,dt\int_0^\infty e^{-u^2}du=I\int_0^\infty\frac{\sin t}{\sqrt{t}}dt=I\sqrt{\frac{\pi}{2}}, \tag{105}$$

by problem 28(a) in 4.12 Exercises of Debnath and Bhatta [4]. Equating the two values of K in (104) and (105) gives

$$I=\int_0^{\infty} e^{-u^2}du=\sqrt{\frac{\pi}{4}}.$$

We use this integral I to evaluate

$$f(x)=\int_0^{\infty} e^{-u^2}\cos xu\,du.$$

It is easy to check that $f(x)$ satisfies the differential equation $f'(x)+\frac{x}{2}f(x)=0$ and its solution is

$$f(x)=C\,exp\left(-\frac{x^2}{4}\right),\quad \text{where } C=f(0)=\frac{\sqrt{\pi}}{2}.$$

Consequently,

$$f(x)=\frac{\sqrt{\pi}}{2}\,exp\left(-\frac{x^2}{4}\right). \tag{106}$$

In probability theory, the number π also occurs in the binomial probability distribution P of r successes in n trials with the probability p of each success and $q=(1-p)$ of each failure given by

$$P=\frac{n!}{r!(n-r)!}\times p^r\times(1-p)^{n-r}. \tag{107}$$

We write $r=\ np+x, n-r=\ (1-p)\,n-x\ =nq-x$. Since n is large , r is also large provided x is small compared with np. We use $Stirling's$ approximation formula and expand $log\,P$ in descending powers of n so that

$$\begin{aligned}log\ P&=log\,n\,!+(np+x)\,log\,p+(qn-x)\,log\,q-log\,(np+x)\,!-log\,(nq-x)\,!\,.\\&=-\frac{1}{2}\,log\,(2\pi pqn)-\frac{1}{2n}\left[\frac{x^2}{pq}+\frac{x(1-2p)}{pq}\right]\cdots.\end{aligned}$$

Thus,

$$P\backsim\frac{1}{\sqrt{2\pi pqn}}exp\left[-\frac{1}{2n}\left\{\frac{x^2}{pq}+(1-2p)x/pq\right\}\right].$$

If $|x|>>|1-2p|$, we can neglect the term $(1-2p)x$ compared with x^2, and obtain the approximate result for the probability P of the binomial distribution which involves $\sqrt{\pi}$ as

$$P\backsim\frac{1}{\sqrt{2\pi pqn}}exp\left(-\frac{x^2}{2npq}\right). \tag{108}$$

We next consider the Tchebycheff problem dealing with the probability of two integers in the range 2 to n are primes to each other. Any number when divided by a prime factor r may have a remainder 0,1,..., $r-1$, hence, the probability that it is divisible by r is $\frac{1}{r}$. Thus both integers are divisible by r is $\frac{1}{r^2}$, so that the both integers are *not divisible* by r is $\left(1-\frac{1}{r^2}\right)$. Thus, the probability that two integers have *no* common prime factor in the whole range is

$$x=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{p^2}\right), \tag{109}$$

where p is the greatest prime in (2,n). If n (and hence p) is large , the approximate value x is given by the infinite product

$$x = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{r^2}\right)\cdots, \tag{110}$$

where r is a prime number.

Thus,

$$\begin{aligned}\frac{1}{x} &= \left(1 - \frac{1}{2^2}\right)^{-1}\left(1 - \frac{1}{3^2}\right)^{-1}\cdots\left(1 - \frac{1}{r^2}\right)^{-1}\cdots \\ &= \left(1 + \frac{1}{2^2} + \frac{1}{4^2} + \cdots\right)\left(1 + \frac{1}{3^2} + \frac{1}{3^4}\cdots\right)\end{aligned}$$

Since any number is either a prime or a product of primes, it follows by multiplying out that

$$\frac{1}{x} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}.$$

Hence,

$$x = \frac{6}{\pi^2}. \tag{111}$$

Conversely, given x, the Tchebycheff solution (111) can be used to find an approximation of π. For, example the probability x that the number of pairs between 2 and 30 has no common factors is

$$\begin{aligned}x &= \frac{\text{Total number of pairs between 2 and 30 has no common factors}}{\text{Total number of pairs between 2 and 30}} \\ &= \frac{248}{29_{C_2}} = \frac{248}{406} = \frac{6}{\pi^2}, \quad \text{by (111).}\end{aligned}$$

Thus, $\pi^2 \backsim 9.823$, or, $\pi \backsim 3.1342$.

We must consider the approximate value of Bernoulli's probability in a given population of coins which has equal number of heads and tails. If there are (n + r) heads and (n − r) tails in large sample of $2n$, the probability p of a head or tail is $\frac{1}{2}$ and Bernoulli's probability gives the formula

$$P = \binom{2n}{n+r}\left(\frac{1}{2}\right)^{n+r}\left(1 - \frac{1}{2}\right)^{n-r} = \frac{(2n)!}{(n+r)!(n-r)!} \times \frac{1}{2^{2n}} \tag{112}$$

Using the well-known Stirling approximation formula for $n!$, (112) can be simplified as

$$P = \frac{1}{\sqrt{\pi n}}\left(1 - \frac{r^2}{n^2}\right)^{-n}\left(1 - \frac{r}{n}\right)^{r-\frac{1}{2}}\left(1 + \frac{r}{n}\right)^{-r-\frac{1}{2}} \tag{113}$$

which is, by $(1-p)^n \backsim e^{-np}$,

$$P \backsim \frac{1}{\sqrt{\pi n}} e^{\frac{-r^2}{n}} \backsim \frac{1}{\sqrt{\pi n}} \tag{114}$$

provided $\frac{r}{n}(< 1)$ is small.

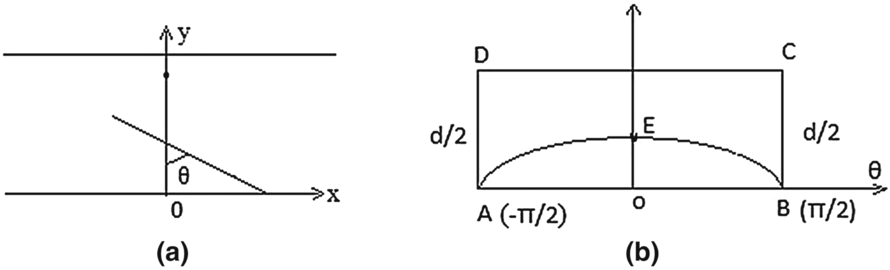

Fig. 3 (**a**), (**b**) The Buffon needle problem

In a sample of size $2n$, the probability that the number of heads will lie between $n+s$ and $n-s$ is approximately given by

$$P_s = \sum_{r=-s}^{s} \frac{1}{\sqrt{\pi n}} e^{-\frac{r^2}{n}},$$

where $s = 0, 1, 2, \cdots$, s.

To evaluate P_s, we write $r = x\sqrt{n}$, where the increment of r is one and $(r+1) = (x+\delta x)\sqrt{n}$ so that $\delta x = \frac{1}{\sqrt{n}}$. Thus,

$$P_s = \sum_{x=-\frac{s}{\sqrt{n}}}^{\frac{s}{\sqrt{n}}} \frac{\delta x}{\sqrt{\pi}} e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{\frac{-s}{\sqrt{n}}}^{\frac{s}{\sqrt{n}}} e^{-x^2} dx = \frac{2}{\sqrt{\pi}} \int_0^{\frac{s}{\sqrt{n}}} e^{-x^2} dx = erf\left(\frac{s}{\sqrt{n}}\right),$$

where the error function erf (x) represents the probability integral.

We next consider the Buffon needle problem (see Fig. 3(a), (b)). A paper is ruled with parallel lines at a distance d apart. A needle of length $l \le d$ is thrown at random on the paper. What is the probability that the needle will intersect with the lines?

We take one of the parallel lines as the x-axis and the normal to it as y-axis. The probability that the center of the needle has an ordinate lying between y and $y + dy$ *is* $\frac{\mathrm{dy}}{d}$, and the probability that the inclination θ of the needle to the y-axis between θ and $\theta + d\theta$ is $\frac{\mathrm{d}\theta}{\pi}$. Thus, the probability p that the needle will cross the x-axis is given by the double integral

$$p = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\pi} \int_{-\frac{\ell}{2}\cos\theta}^{\frac{\ell}{2}\cos\theta} \frac{\mathrm{dy}}{\mathrm{d}} = \frac{2}{\pi d} \int_0^{\frac{\pi}{2}} (\ell\cos\theta)\, d\theta = \frac{2\ell}{\pi d}. \tag{115}$$

Alternatively, the possible values of y are $|y| = \frac{1}{2} l \cos\theta$, where $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$, which represents the curve AEB and AD = BC = $\frac{1}{2}$d.

The probability that the needle will intersect one of the parallel lines is

$$p = \frac{\text{Area } AEB}{\text{Area } ABCD} = \frac{2\triangle OEB}{\frac{\pi d}{2}} = \frac{4}{\pi d} \int_0^{\frac{\pi}{2}} \frac{\ell}{2} \cos\theta d\theta = \frac{2\ell}{\pi d}. \tag{116}$$

Thus, the Buffon needle problem in probability and statistics involves the constant π.

Finally, if n is the number of elements in any sample of magnitude T belonging to a subclass A, then the probability of an element $\in$ A is $\frac{1}{n}$, and hence, the probability that it does *not* belong to A is $\left(1 - \frac{1}{n}\right)$. The number of elements in the sample of total magnitude t is $\frac{nt}{T}$, and the probability that none of these elements belongs to A is given by

$$P = \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n}\right) \cdots \frac{nt}{T} \text{ times} = \left(1 - \frac{1}{n}\right)^{\frac{nt}{T}} = \left[\left(1 - \frac{1}{n}\right)^{-n}\right]^{\left(-\frac{t}{T}\right)}.$$

If n is sufficiently large (n$\rightarrow \infty$), the required probability is the limiting value

$$P = \left\{\lim_{n\to\infty}\left[\left(1 - \frac{1}{n}\right)^{-n}\right]\right\}^{-\frac{t}{T}} = e^{-\frac{t}{T}}. \tag{117}$$

If we apply this result to the famous example that the probability that the Sun *will not rise* tomorrow is e^{-1}, and hence, the probability that the Sun *will rise* tomorrow at least once is $\left(1 - e^{-1}\right) \approx 0.6322$. This may not be a real answer.

The Euler Quadratic Polynomial and the Gauss Class Number Conjucture

In order to generate prime numbers, Euler discovered a remarkable quadratic polynomial $f(n) = n^2 + n + 41$ for integer n. For $n = 0$ to 39, $f(n)$ is a prime number. For example, $f(0) = 41$, $f(1) = 43$, $f(2) = 47$ *and* so on. Thus, the Euler quadratic function produces a large number of prime numbers, but $f(40) = 41^2 = 1681$ which is not a prime. No other quadratic expression has been discovered which produces prime numbers. So, the question is whether primes could be generated as the values of a polynomial function ? In 1970, the 22-year old Russian mathematician, Yuri Matiyasevich proved merely the existence of a prime generating polynomial function what Euler did about 300 years ago . Since the problem is beyond the scope of this article, we will not pursue this problem further.

It is amazing that the roots of the Euler quadratic equation $x^2 + x + 41 = 0$ have a remarkable property (see Debnath [1]). The roots are complex numbers given by

$$x = \frac{1}{2}\left(-1 \pm i\sqrt{4 \times 41 - 1}\right) = \frac{1}{2}\left(-1 \pm i\sqrt{163}\right). \tag{118}$$

Like e, $\sqrt{163}$ is an irrational number and like $e^{i\pi}$, $\exp(\pi\sqrt{163})$ is close to an integer which is calculated up to twelve decimal places as

$$e^{\pi\sqrt{163}} = 262 \quad 537 \quad 412 \quad 640 \quad 768 \quad 744.000 \quad 000 \quad 000 \quad 000. \tag{119}$$

A more accurate value of $\exp(\pi\sqrt{163})$ upto fifteen decimal places is

$$262 \quad 537 \quad 412 \quad 640 \quad 768 \quad 744.999 \quad 999 \quad 999 \quad 999 \quad 250. \tag{120}$$

which is incredibly close to an integer in (119), whereas for most other natural numbers k the value of $\exp(\pi\sqrt{k})$ is not very close to an integer. So, in some sense the number 163 has the special property. The question is why the number 163 is so special? The reason is associated with Gauss class number problem so that 163 is the largest value of d for which the number system $(a + i\sqrt{d})$ has the unique factorization property. With each number system derived from some value of d, Gauss identified a certain natural number h(d) called the *class number* of that system. Based on extensive computation, he also observed that for each class number k, there exists a largest value of d for which $h(d) = k$. He also found nine values of $d = 1, 2, 3, 7, 11, 19, 43, 67,$ and 163 for $h(d) = 1$ so that the largest d was 163. This explained why the number $163 = 4 \times 41 - 1$ was so special in Euler's quadratic equation. On the other hand, the largest d for which $h(d) = 2$ seemed to be $d = 427$ and the largest d with $h(d) = 3$ was $d = 907$. Gauss was neither able to confirm that any of these values really was the largest, nor proved that there always was a largest d. So, this became one of the *Gauss conjectures*. In 1952, this class number conjecture h(d) = 1 for d = 163

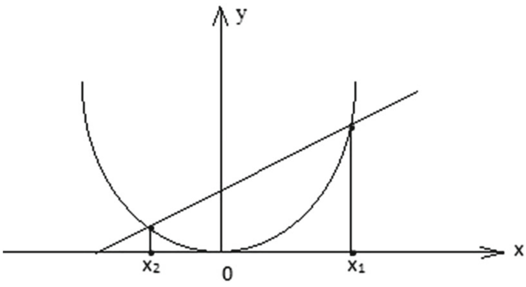

Fig. 4 Intersection of y = x + 1 and y = x^2

was solved by a retired Swiss mathematician, Kurt Heegner (1893–1965). However, nobody believed his proof because his paper was difficult to understand. Another fifteen years later in 1967, Harold Stark of MIT and Alan Baker of the University of Cambridge, England provided independently different proofs to establish that there is no tenth d. So, the Gauss class number conjecture is solved.

The Golden Number *g*

From ancient times, the algebraic method of solution of the equation $x^2 = x + 1$ can be transformed into geometrical problem of finding the intersection of the line $y = x + 1$ and the parabola $y = x^2$ as shown in Fig. 4.

Using the classical formula of the quadratic equation, two solutions of the equation $x^2 = x + 1$ are

$$x_1 = \frac{1+\sqrt{5}}{2} \approx 1.618, \tag{121a}$$

$$x_2 = \frac{1-\sqrt{5}}{2} = -0.618. \tag{121b}$$

Each of the points (x_1, y_1) *and* (x_2, y_2) where the line intersects the curve $y = x^2$ are x_1 and x_2 which represent the two possible solutions $x = x_1$ *and* $x = x_2$.

We next solve the same equation $x^2 = x + 1$ *or* $x = \sqrt{x+1}$ by using a table. Putting $x + 1 = t$ *so that* $t - 1 = \sqrt{t}$, that is, to find a number whose square root is one less than the number itself. We then generate a short table as

$t =$	2.617,	2.618	2.619	2.620, ...
$\sqrt{t} =$	1.617 71	1.618 0234,	1.618 3324,	1.618 6414, ...

The solution is $\sqrt{x+1} = \sqrt{t} = 1.618$ as stated above. Evidently, the graphs of y = $\sqrt{t}$ and y = t−1 would intersect at t = 1.618. It is noted that the table does not give an exact solution, but it is the nearest that can be obtained from the table.

In Greek mathematics, this remarkable number $x_1 = \frac{1}{2}(1 + \sqrt{5})$, denoted by g or τ , is called the *golden number (or golden ratio)* and is defined in geometry by dividing a line segment in such a way that the ratio of the total length ℓ to the larger segment x is equal to the ratio of the larger to the smaller segment. In other words, the golden ratio $g = \frac{\ell}{x}$ is determined by the equation

$$\frac{\ell}{x} = \frac{x}{\ell - x}. \tag{122}$$

Or, equivalently,

$$g^2 - g - 1 = 0. \tag{123}$$

The positive solution of the quadratic equation (123) is

$$g = \frac{\ell}{x} = \frac{1}{2}\left(\sqrt{5}+1\right) = 1.618. \tag{124}$$

The inverse ratio of g is

$$\frac{1}{g} = \frac{x}{\ell} = \frac{1}{2}\left(\sqrt{5}-1\right) = 0.618, \tag{125}$$

so that $\frac{1}{g} = g - 1$. There is an interesting relation between g and π

$$\frac{6}{5}(g+1) = \frac{6g^2}{5} \approx \frac{6}{5}(g+1) = \frac{6}{5}(2.6180339) = 3.1416406 \approx \pi. \tag{126}$$

It follows from the golden quadratic equation that

$$g^2 = (\sqrt{g})^2 + 1^2 \tag{127}$$

which means that $(1, \sqrt{g},\ g\ )$ form a golden right angled triangle. It is then easy to verify that

$$g^n(1, \sqrt{g}, g) = \left(g^n, g^{n+\frac{1}{2}}, g^{n+1}\right) \tag{128}$$

is also a golden right angled triangle because

$$(g^n)^2 + (g^{n+\frac{1}{2}})^2 = g^{2n} + g^{2n+1} = g^{2n}(g+1) = g^{2n}g^2 = (g^{n+1})^2. \tag{129}$$

All the interested readers are referred to a recent article of Debnath [5] which dealt with elaborate properties of the golden numbers with numerous applications.

The Feigenbaum Number δ

Historically, in his *De Analyst* and *Method of Fluxions*, Sir Isaac Newton formulated a general classic algorithm in 1669 for approximating real roots of an equation $f(x)$=0, which was published in Wallis's *Algebra* of 1685. However, the idea was known to the ancient Babylonians for approximating the roots of quadratic equations. In his tract, *Analysis Aequationum Universalis* (1690), Joseph Raphson (1648–1715) made some improvement on the Newton method, although he applied it only to polynomial equations, but it was much more useful for other equations. It is this modification is now known as the *Newton method* or the *Newton-Raphson method*. The fundamental idea in Newton's method is to use the tangent line approximation of the function $y = f(x)$ at a chosen point x_1, called the *first approximation* of the root α of f(x)=0. Thus, the equation of the tangent line to $y = f(x)$ *at* $(x_1, f(x_1))$ is given by

$$y - f(x_1) = f'(x_1)(x - x_1), \qquad f'(x_1) \neq 0. \tag{130}$$

When y=0, the x-intercept $x = x_2$ of line (130) is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{131}$$

which is called the *second approximation* of the root α as x_2 is much closer to α. Proceeding similarly, we obtain the *third approximation* x_3 given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}, \qquad f'(x_2) \neq 0. \tag{132}$$

In general, the $(n+1)th$ *approximation* of the root α is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0. \tag{133}$$

The numbers $x_1, x_2, \cdots x_n, \cdots$ are called successive approximations of the root α. If $\{x_n\}$ becomes closer and closer to α as $n \to \infty$, then $\{x_n\}$ converges to α. The rapid convergence makes the Newton method very useful for numerical computations.

If the Newton method is applied to the equation $f(x) = x^2 - a = 0$, then (133) reduces to the arithmetic mean of x_n and $\left(\frac{a}{x_n}\right)$, that is,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \tag{134}$$

where $n = 1, 2, 3, \cdots$. This algorithm was known to Heron of Alexandria (10-50 AD) for approximating square roots of a number and was also used by the ancient Babylonians for approximating the roots of quadratic equations.

As $n \to \infty$, $x_n \to \alpha = \sqrt{a}$. In this case, convergent occurs for any $x_1 > 0$. We can write (134) in the form

$$x_{n+1} = g(x_n), \tag{135}$$

where $g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$ *so* that $f(\alpha)$=0 implies $g(\alpha)$=α which means that α is a *fixed point* of the function g *as* $y = g(x)$ intersects the line $y = x$, whereas $f(\alpha)$=0 means that $y = f(x)$ crosses the x-axis ($y = 0$). In general, the roots and fixed points (if they exist) are *not* the same. For example, $y = f(x) = x^2 - \frac{1}{2}$, the roots of $f(x) = 0$ are $x = \pm\frac{1}{\sqrt{2}}$, whereas the fixed points of $f(x)$ are solutions $x^2 - \frac{1}{2} = x$, that is , $x = \frac{1}{2}(1 \pm \sqrt{3}) \approx 1.3660$ and -0.3660.

If we apply the Newton method to the equation $f(x) = \frac{1}{\mathrm{x}} - a = 0$, then formula (133) gives

$$x_{n+1} = 2x_n - a\, x_n^2. \tag{136}$$

Thus, the Newton algorithm enables us to find the reciprocals *without* division. However, the division was used in early days to find the roots of $f(x) = \frac{1}{\mathrm{x}} - a = 0$ so that $x = \frac{1}{a}$, $a \neq 0$.

The generalization of the Newton formula (133) for a complex variable z leads to the form

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \quad f'(z_n) \neq 0. \tag{137}$$

In particular , if $f(z) = z^2 - 1 = 0$, (133) becomes

$$z_{n+1} = \frac{1}{2}\left(z_n + \frac{1}{z_n}\right). \tag{138}$$

With the chosen first approximation z_0 with a positive real part, z_n converges to the positive root, $z = 1$. Similarly, for a negative real part of z_0, z_n converges to the negative root $z = -1$.

However, for purely imaginary $z_1 = iy_1$, where $y_1(\neq 0)$ is real, it does not converge at all. Formula (138) with $z_n = iy_n$ becomes

$$y_{n+1} = \frac{1}{2}\left(y_n - \frac{1}{y_n}\right). \tag{139}$$

For example, $y_1 = g = 1.618$, where g is the golden number, $y_2 = -0.75$, $y_3 = 0.2916$, $y_4 = -1.56845$, $\cdots$ etc. For other value of $y_1 = 1 + \sqrt{2}$, $y_2 = 1$, $y_3 = 0$ *and* $y_4 = \infty$.

Thus, for $f(z) = z^2 - 1$, the simple mapping given by the Newton formula has rather a very strange feature for the initial values on the imaginary axis. More generally, two different indistinguishable initial conditions often lead to an unstable situation as the system evolves in space and time. So, some of both discrete and continuous dynamical systems are extremely sensitive to initial conditions, and hence, they lead to unstable state which is often called *chaos*. For example, water flowing through a circular pipe offers one of the simple models of chaos. When pressure is applied to the end of the pipe, water flows in straight lines. As pressure increases and reaches a critical value, the laminar flow breaks down to a totally new chaotic form of motion known as *turbulence*. Thus, a slowly simple laminar flow suddenly changes to a turbulent flow of beautiful and complex form consisting of highly irregular random motions of self-similar small eddies of all sizes or scales. The path of any fluid particle is quite predictable before the onset of turbulence. Once turbulence occurs, the predictability is completely lost. Thus, chaos deals with situation of a physical system in which a sudden change transforms a predictable state into an unpredictable one.

With a complex number z_n, the complex quadratic polynomial

$$z_{n+1} = z_n^2 - a, \tag{140}$$

represents a remarkable example of chaos. Using an electronic computer, an unimaginably complex but a remarkably beautiful graph of (140) can be constructed. Often, regions of chaotic behavior under magnification lead to a self-similar fractal curve.

Another application of Newton's method to $f(z) = z^3 - 1$ gives

$$z_{n+1} = \frac{\left(2z_n^3 + 1\right)}{3z_n^2} \tag{141}$$

In this case, all z_1 will *not* converge toward the closest of the three roots of $f(z) = 0$. Thus, $z_1 = -1$ converges to 1, and the point z = 0 maps, of course, to ∞. For $|z_1| \leq 1$, $z_2 \approx \frac{1}{3z_1^2}$, a large number and the next approximation makes z smaller again. Thus, it is almost unpredictable about the nature of $\{z_n\}$ as $n \to \infty$.

The family of real *logistic map (or logistic parabola)*

$$f_\lambda(x) = \lambda\, x(1 - x), \quad 0 \leq x \leq 1, \tag{142}$$

is a celebrated example of one-dimensional parameterized family of dynamical systems, where $\lambda \in [0, 4]$ is the parameter. This quadratic map (142) illustrates all general features of nonlinear maps including stability, periodic solutions and bifurcations. Both qualitative and quantitative features of a dynamical system depend on the *parameter*. A parameter value(s) where the qualitative behavior changes is called a *bifurcation value(s)* of the parameter. The fixed points of the logistic map (142) are given by solutions of $f_\lambda(x) = x$ so that the two fixed points are $x = 0$ and $x = \left(\frac{\lambda-1}{\lambda}\right)$. For the family of logistic map (142), the parameter value $\lambda = 3$ is a *bifurcation value* because the stability of the fixed point $x = \left(\frac{\lambda-1}{\lambda}\right)$ changes from repelling to attracting. On the other hand, $\lambda = 1$ is also a bifurcation value because for

$\lambda < 1$, $x = 0$ is the only fixed point, and for $\lambda > 1$, $f_\lambda(x)$ has two fixed points. Moreover, the bifurcation value $\lambda = 3$ is *generic* in the sense that bifurcation occurs for all neighboring families of dynamical systems. When λ is close to 3, the graph of $f_\lambda(x)$ crosses the diagonal transversely at the fixed point $x_\lambda = \left(1 - \frac{1}{\lambda}\right)$, and $|f'_\lambda(x_\lambda)| < 1$ for $\lambda < 3$, and $\left|f'_\lambda(x_\lambda)\right| > 1$ for $\lambda > 3$.

For logistic maps $f(x) = \lambda$x(1−x) (or many other nonlinear maps), the iterated map $(fof)(x) = f^{(2)}(x) = f(f(x))$ is similar to that of the original map, $f(x)$. The same property hold for $f^{(4)}(x),\ f^{(8)}(x),\ \cdots$ *so that* $f^{(2^n)}(x)$ becomes increasingly similar to each other as n increases. So, this leads to a self-similar scaling law for the parameter values λ_n at which bifurcation occurs. Sarkovskii [6] proved, for the frequently used Ritcker's map (see Ritcker [7])

$$x_{n+1} = x_n\, exp[\lambda(1 - x_n)], \tag{143}$$

if a solution of *odd* ($\geq$ 3) period exists for a value $\lambda = \lambda_c$, then aperiodic or chaotic solutions exists for $\lambda > \lambda_c$. Such solutions simply oscillate in a random manner, the larger the parameter λ the larger the amplitude of the oscillatory solution. Subsequently, Stefan [8] generalized the Sarkovskii result. Li and York [9] proved that, if a solution of period 3 exists, then solutions of period n exist for all $n \geq 1$.

If $1 < \lambda < 3$, (142) has an unique nontrivial limit $x*$. Any $x_0 \neq 0$ or 1 will be attracted to x*. For example, $\lambda = 2$, the attractor $x*$ is equal to $x* = \frac{1}{2}$. With $x_0 = 0.8$, (142) gives a set of values, $x_1 = 0.3200, x_2 = 0.4352, x_3 = 0.4916, x_4 = 0.4999, x_5 = 0.4999$ which converges to $x* = \frac{1}{2}$.

On the other hand, if $\lambda > 3$, the attractor becomes *unstable*. For $\lambda = 3.2$ and $x_0 = 0.8$, (142) gives another set of values, $x_1 = 0.5120, x_2 = 0.7995, x_3 = 0.5130, x_4 = 0.7995, x_5 = 0.5130, x_6 = 0.7995, x_7 = 0.5130, \cdots$. There is no single-valued attractor, and x* has bifurcated into an orbit of period 2. In other words, there is a periodic attractor characterized by two values of x = 0.5130, and 0.7995 and other nontrivial values of x_0 are attracted to this pair of values.

In 1978, based on numerical computations, Mitchell Feigenbaum ([10,11]) observed that, for bifurcation values λ_1=3, $\lambda_2 = 3.449\,499,\ \lambda_3 = 3.544\,090,\ \lambda_4 = 3.564\,407,\ \lambda_5 = 3.568\,891,\ \lambda_6 = 3.569\,692,\ \lambda_7 = 3.569\,891,\ \lambda_8 = 3.569\,934,\ \cdots$ with their consecutive increases $(\lambda_2 - \lambda_1),\ (\lambda_3 - \lambda_2),\ (\lambda_4 - \lambda_3),\ (\lambda_5 - \lambda_4),\ (\lambda_6 - \lambda_5),\ (\lambda_7 - \lambda_6),\ (\lambda_8 - \lambda_7),\ \cdots$ so that the *ratios* of consecutive increases are

$$F_n = \frac{(\lambda_n - \lambda_{n-1})}{(\lambda_{n+1} - \lambda_n)}, \quad n = 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ \cdots. \tag{144}$$

In particular, the sequence $F_2 = 4.7520\,271, F_3 = 4.6557\,562, F_4 = 4.6684\,283, F_5 = 4.6645\,231,\ F_6 = 4.6884\,422,\ F_7 = 4.6279\,069, \cdots$ approaches to a certain limit whose accurate value can be obtained by a modern computer in the form

$$\lim_{n\to\infty} F_n = 4.6692\,016\,099\cdots \quad = \delta, \tag{145}$$

where δ is called the *Feigenbaum universal constant.*

If the nonlinear parameter λ increases, the attractor will bifurcate again and have period 4=2^2. Similarly, further increase of λ will lead to periods $2^3,\ 2^4,\ 2^5\ \cdots$ and so on, until at a critical value λ_∞, the period will become infinity. For such a period doubling bifurcation phenomenon with eventually leading to chaotic solutions, if λ_n represents the value of λ at the nth bifurcation, then the limit of the successive parameter intervals as n$\to\infty$ converges to the celebrated *Feigenbaum constant* δ. More precisely,

$$\delta = \lim_{n\to\infty}\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n}\right) = 4.669\ 201\ 6099\cdots. \tag{146}$$

This is now known as the *Feigenbaum universal constant* 4.6692…which often occurs in many problems in nonlinear dynamics. Mitchell Feigenbaum originally discovered it in 1978 using the period doubling bifurcations in his study of the quadratic family of maps $f_\lambda(x) = 1 - \lambda\, x^2$ in $-1 \le x \le 1$ for $0 < \lambda \le 2$. For $\lambda < \frac{3}{4}$, he found the unique attracting fixed point $x_\lambda = \frac{1}{2\lambda}[\sqrt{1+4\lambda} - 1]$ of $f_\lambda(x)$. The derivative $f'_\lambda(x)$ at $x = x_\lambda$ is $(1 - \sqrt{1+4\lambda})$ which is greater than -1 when $\lambda < \frac{3}{4}$, is equal to -1 when $\lambda = \frac{3}{4}$ and is less than -1 when $\lambda > \frac{3}{4}$. Thus, when $\lambda > \frac{3}{4}$, the map $f_\lambda(x)$ has an attractive periodic orbit of period 2. His numerical computations reveal that there is an increasing sequence of bifurcation values λ_n at which an attractive periodic orbit of period 2^n for $f_\lambda(x)$ loses stability, and an attractive periodic orbit of period 2^{n+1} appears. This led to the discovery of the same Feigenbaum constant δ defined by (146). Many numerical computations prove the existence of the Feigenbaum constant δ for several other one-parameter families of logistic maps. He also showed that it is related to almost one-dimensional nonlinear maps with a single quadratic maximum. So, this general nature of this universal constant reveals that every chaotic system possesses a similar bifurcation phenomenon. In 1977, Grossmann and Thomas [12] independently discovered the same numerical constant δ. It is noted that it is a mathematical quantity , but it was discovered by mathematical physicists, and most of the papers have been published in the physics literature. Later on, mathematicians contributed to the subsequent development of the theory. The Feigenbaum constant in bifurcation theory is analogous to the Archimedes constant π in geometry and the Euler constant e in calculus. Like π and e, δ is believed to be transcendental, but it has not yet been proved or disproved. Feigenbaum also discovered the second constant $\alpha = 2.5029$ which is also believed to be transcendental and it also arises in dynamical system, but it is not as well known like δ.

Like π or e, there is no closed form exact formula or infinite series available which can be used to calculate δ, and α, but there exist closed form approximations. One of the most accurate approximations is

$$\pi + \tan^{-1}\left(e^{\pi}\right) \approx 4.669202. \tag{147}$$

There are also two closely related formulas which accurately approximate both δ and α as

$$\delta = \frac{2g}{\log_e 2} \approx 4.6692\cdots, \tag{148a}$$

$$\alpha = \frac{2g+1}{\log_e 2 + 1} \approx 2.5029\cdots, \tag{148b}$$

where g= 1.618 is the golden number.

The Ritcker map (143) can be used to calculate the Feigenbaum constant δ (see McCartney and Glass [13]) in 21 decimal places

$$\delta = 4.6692\ 0160\ 9102\ 9906\ 7185\ 3\ \cdots. \tag{149}$$

It is also important to point out that the logistic map becomes even more interesting if the parameter λ is allowed to assume complex values. In fact, the computer graphs of the complex logistic maps reveal a fractal image with self-similar properties. In a nutshell, the simple logistic model has some special properties to describe many interesting and unexpected features of nonlinear dynamical systems.

In his pioneering work on modern differential equations, the last universal mathematical scientist, Henri Poincaré (1854–1912) first discovered some remarkable qualitative features of a dynamical system. A discrete (or continuous) dynamical system is a system to describe the way the system evolves in discrete (or continuous) time. Poincaré also first observed the sensitivity of initial conditions in his celebrated study of a three-body problem in celestial mechanics. It is almost impossible to make a long time prediction of a dynamical system that exhibits sensitivity of initial data which play an important role on the idea of the so-called *strange attractors*. A set of points in a set A is called an *attractor* if all orbits (or trajectories) that start in A remain in A and all neighboring orbits get closer and closer to A. A set of points $x_1, x_2, \cdots, x_n$ is obtained by a process of iteration from $x_{n+1} = f(x_n)$ with a starting point x_0. The number x_n is called the *orbit* of x_0, and it is called a *periodic orbit* of period n if it repeats itself after time n, but not earlier. A discrete dynamical system is called *deterministic* if the orbit of any initial point x_0 is completely determined once x_0 is known. A periodic orbit of x_0 is called *stable* if all orbits remain close to it at all future times. It is called *asymptotically stable* if all sufficiently close orbits tend to it as time $t \to \infty$. The long time behavior of such systems is of special interest. Two standard examples of discrete systems include (i) approximation of roots by the celebrated *Newton-Raphson method*, and (ii) the famous *discrete logistic* model, which have been discussed above in some detail.

There is also wide spread interest in continuous dynamical systems described by ordinary differential equations which often exhibit chaotic behavior. In his 1963 seminal paper [14], Edward Lorenz (1917–2008) discovered a system of three nonlinear ordinary differential equations for the three state variables x(t), y(t), z(t) in the form

$$\frac{dx}{dt} = \sigma(y - x), \tag{150a}$$

$$\frac{dy}{dt} = rx - (y + xz), \tag{150b}$$

$$\frac{dz}{dt} = xy - bz, \tag{150c}$$

where σ, r, and b are parameters. These equations are known as the *Lorenz system of equations* whose numerical solutions for certain values of parameters and initial conditions exhibit chaotic behavior. Usually, if the value of the parameter r is varied with two parameters σ and b held constant, a wide variety of dynamical behaviour is observed. One numerically computed solutions of the Lorenz system with a plot of z against x with $\sigma = 10, r = 28, \ b = \frac{8}{3}$ and $(x(0), \ y(0), \ z(0)) = (5, \ 5, \ 5)$ is shown in the Fig. 5 which resembles a *butterfly* or *figure eight*. When such solutions are plotted for different values of r, they typically look like Fig. 5. This is a sample example of the so called *Lorenz attractor* for $\sigma = 10, \ r = 28 \ and \ b = \frac{8}{3}$.

His numerical experiments show that the Lorenz system captured a certain new but intrinsic chaotic behavior of the weather phenomena, and the long time weather prediction is almost impossible. As the parameters are varied, the solutions of the Lorenz differential system exhibit bifurcations to higher periodic solutions eventually leading to chaotic (or aperiodic) behavior. It also revealed that a very small change in initial conditions leads to a significantly large change in the solution.

It is now a well-known fact that a temporal behavior of the Lorenz system is stochastic in nature. In fact, the short time behavior is *deterministic*, but its long time evolution is *stochastic* in nature. This phenomenon is known as the *deterministic chaos*. Indeed, many nonlinear dynamical systems have some special features similar to those of Lorenz attractors as shown in Fig. 5.

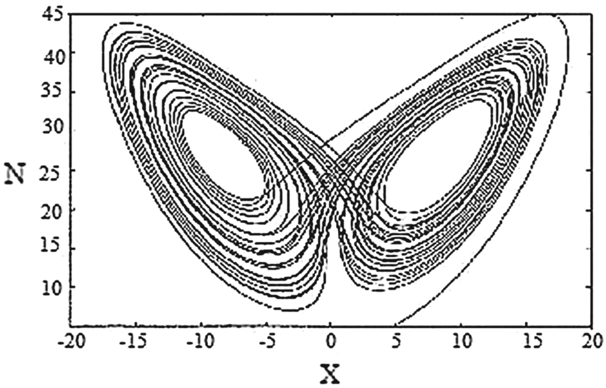

Fig. 5 A numerically computed typical solution of the Lorenz system with a *plot* of z against x with r = 28, (x(0), y(0), z(0) = (5,5,5,))

In general, modern numerical computations reveal that the period-doubling bifurcation occurs for large values of the parameter r. At each period-doubling bifurcation, the original periodic orbit becomes unstable, and new stable orbits which wind twice as much as the original will appear. For such period-doubling phenomena, if r_n represents the value of r at the nth bifurcation, then, in the limit as $n \to \infty$, the celebrated *Feigenbaum constant* δ appears which is given by

$$\delta = \lim_{n\to\infty}\left(\frac{r_n - r_{n-1}}{r_{n+1} - r_n}\right) = 4.669\,201\,6609\cdots . \tag{151}$$

Another distinctive feature of the Lorenz attractor is that it has *non-integral* (or *fractal*) *dimension*, $D_L = 2.06$ ($2.00 < D_L < 2.401$). In view of fact that the Lorenz attractor is a set of chaotic solutions of the Lorenz system with fractal diemension, it is called a *strange attractor*. In fact, the Lorenz attractor is the best known as the *strange attractor*. In 1970s, the concept of chaos arose as the revelation of the Lorenz (strange) attractor. The chaotic motion is found to be very irregular, and it appears to be *random* in nature, but it is deterministic. Physically, chaotic form of motion known as turbulence in fluid dynamics where the slowly moving fluid particles (laminar flow) become rapidly moving turbulent flow consisting of highly complex irregular motions of self-similar small eddies of various sizes or scales (see Debnath [15]) and it often arises in many problems in nonlinear dynamics. Recent modern computer simulations have confirmed the above features of the Lorenz attractor. But the problem is whether they can be used as a mathematical proof of such sensitive dynamical systems, and it has become an open problem for a short time. In 2002, Warwick Tucker [16] used the interval arithmetic method where a number is represented by an interval to develop a computer-aided rigorous proof that the Lorenz attractor is a strange attractor. Thus, the problem is solved.

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